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**DISEÑO ROBUSTO APLICADO A LA
EXPERIMENTACIÓN AGRÍCOLA Y A
PROCESOS AGRO-INDUSTRIALES**

Alphonse BIZIMANA

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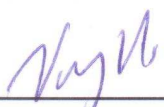
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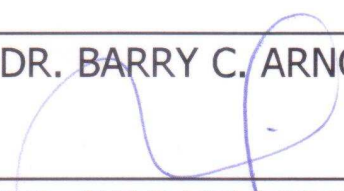
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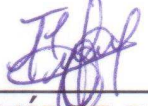
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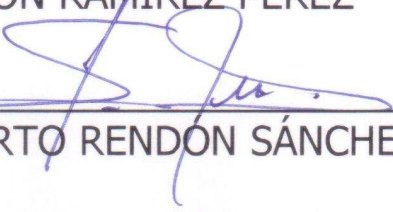
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Dedication

To my family

Esta tesis corresponde a los estudios realizados con una beca otorgada por la Secretaría de Relaciones Exteriores del Gobierno de México.

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Diseño Robusto Aplicado A La Experimentación Agrícola Y A Procesos Agro-Industriales

Alphonse BIZIMANA

Colegio de Postgraduados, 2015

Proporcionamos una metodología para planear y llevar a cabo el diseño robusto y analizar los datos obtenidos del experimento. Los métodos que presentamos son el enfoque de Taguchi comúnmente conocido como diseño doble arreglo ortogonal, y el diseño arreglo combinado.

El enfoque de Taguchi consiste en un diseño ortogonal doble, uno para los factores de diseño y otro para los factores de ruido.

El diseño arreglo combinado pone ambos tipos de factores en un diseño. Este diseño permite el análisis de las interacciones entre los factores de diseño y de ruido, y reduce el número de corridas requerido para llevar a cabo un experimento. El análisis de los datos obtenidos a partir de este diseño consiste en ajustar un modelo de regresión en términos de factores de diseño y factores de ruido. Desde el modelo ajustado, se obtienen dos superficies de respuesta, una superficie para la media de la característica de calidad y otra para su varianza.

Los problemas de optimización se basan en el criterio del error cuadrático medio y la función de deseabilidad.

Basado en la función de pérdida cuadrática, evaluamos el impacto económico del diseño robusto para reducir el costo de producción.

Considerando las principales clases de posibles respuestas, estos son, *el valor nominal es mejor, entre más grande mejor y entre más pequeño mejor*, proponemos una prueba estadística para comparaciones por pares de tratamientos con respecto a la razón señal-ruido.

Palabras clave: Diseño robusto, diseño doble arreglo ortogonal, diseño arreglo combinado, metodología de superficie de respuesta, error cuadrático medio, función de deseabilidad, función de pérdida cuadrática, razón señal-ruido.

Robust Design Applied to Agricultural Experimentation and Agro-industrial Processes

Alphonse BIZIMANA

Colegio de Postgraduados, 2015

We provide a methodology for conducting robust design and analysing the data obtained from the experiment. The methods we expose are the Taguchi approach commonly known as double orthogonal array design or crossed array design, and the combined array design.

The Taguchi approach consists of a double orthogonal array design, one for the design factors and another for the noise factors.

The combined array design puts both types of factors in one design. This design permits the analysis of interactions between the design and noise factors, and reduces the number of runs required to conduct an experiment. The analysis of data obtained from this design consists of adjusting a regression model in terms of design factors and noise factors. From the adjusted model, two response surfaces are obtained, one for the mean of the quality characteristic and another for its variance.

The optimization problems are based on the mean square error criterion and desirability function.

Based on quadratic loss function, we assess the economic impact of robust design in reducing the production cost.

Considering the main classes of possible responses, these are *the nominal-the-best*, *the larger-the-better* and *the smaller-the-better*, we propose a statistical test for pairwise comparisons of treatments with regard to the signal-to-noise ratio.

Key words: Robust design, double orthogonal array design, combined array design, response surface methodology, mean square error, desirability function, quadratic loss function, signal-to-noise ratio.

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Chapter 1

Introduction

1.1 Background information

The use of designed experiments plays an important role in several fields such as Economy, Agro-industry and Medicine.

Since engineers and scientists have become increasingly aware of the benefits of using designed experiments, there have been many new areas of application. One of the most important is in robust design. Robust design methodology is a systematic effort to achieve insensitivity to noise factors.

The assumption in robust designs is that there are two types of factors that affect the quality characteristic. These are the control factors and the uncontrollable or difficult to control factors. They are respectively referred to as design factors and noise factors.

Noise factors can be further divided into two categories: external noise factors and internal noise factors. External noise factors are those sources of variability that come from outside of the system. Examples of external noise factors are environmental factors that a system is subject to, such as ambient temperature, ambient pressure and humidity. Internal noise factors are essentially from the variations of control factors. Internal noise could include deviations from the target values of control factors caused by manufacturing, assembly, and deterioration.

While designing an experiment, it is frequently impossible or very difficult or expensive to control or eliminate sources of variation due to external noise factors. However, the experimenter has some control on setting the levels of internal noise factors during the design. The goal of robust design is to enable the experimenter to choose the levels of the control factors that optimize a defined response while minimizing the variation imposed on the process via the noise factors ([Timothy et al. \(2000\)](#)).

[Meng et al. \(2010\)](#) state that robust design is mainly composed of three stages: robust system

1.1. Background information

design, robust parameter design and robust tolerance design. Robust system design consists of using physics, mathematics, experience and knowledge gained in a specific field to develop and select the most appropriate conditions of the design. Once the configuration of a system is finalised, the settings of the nominal levels and the corresponding tolerances need to be determined. Robust parameter design aims at finding the optimal settings of control factors so that the system is insensitive or less sensitive to noise factors. Robust tolerance design is a balancing process. It aims to find the optimal settings of tolerances of the control factors so that the total cost of the system is minimal (Meng et al. (2010)).

The formalization of robust design was initiated by Genichi Taguchi. He introduced an approach referred to as robust parameter design. His approach is based on classifying the factors as either controllable or noise factors, and then finding the settings for the controllable factors that minimize the variability transmitted to the response from the noise factors. Taguchi proposed the use of a cross-product of two experimental designs, known as product array design or double orthogonal array design (Montgomery (2001)). This consists of an inner array containing the design factors and an outer array containing the noise factors. For each combination of design factors, the same array of noise factors is run. The metrics used by Taguchi for evaluating the robustness of a design are the Quadratic Loss Function and the Signal-to-Noise Ratio.

Experimental strategy and design advocated by Taguchi have been criticized by various authors. The main criticism is that the double array design involves a large amount of runs and does not consider interactions between design and noise factors. Details of discussion and criticism of Taguchi's approach can be found in Kunert et al. (2007) and Nair (1992). However, Bisgaard (2000) opened the discussion and the investigation about interactions between the design and noise factors.

As an alternative to the method proposed by Taguchi, other statisticians such as Wu and Hamada (2009) have adopted the use of combined array design which contains controllable and noise factors. This method permits the analysis of interactions between both kinds of factors.

The designs commonly used in conducting double orthogonal array design or combined array design are factorial and fractional factorial designs (Khuri and Mukhopadhyay (1997)). Box and Jones (1992) have introduced the use of split-plot design for robust experimentation.

Response surface methodology is an excellent tool for analysing the data obtained from combined array design. This is a collection of mathematical and statistical techniques that are useful for modeling and analysing problems in which a response of interest depends on several variables, and the objective is to optimize this response.

The statistical methodology underlying robust design, that has by now become the most widely accepted, is the dual response surface methodology which estimates two surfaces, one

1.2. Objectives

for the mean and one for the variance of the quality characteristic ([Giovagnoli and Romano \(2008\)](#)). A modification has been introduced. This modification is related to the division of the noise factors within two groups. The first group consists of random factors and the second is composed of the noise factors for which the levels are fixed. The experiment with the computer is simulated. From the data obtained by simulation, the mean and the variance are calculated. Details of the double orthogonal array design and combined array design are in [Bizimana \(2010\)](#).

When the design scope is extended to the specification of allowable deviations of parameters from the nominal settings (tolerances), the integrated parameter and tolerance design problems arise. The additional objective of minimizing the production costs needed to fulfil tolerance specifications will compete with the minimum variance objective ([Giovagnoli and Romano \(2008\)](#)).

1.2 Objectives

1.2.1 General objective

The main objective of the research is to develop a methodology for improving the study of designed experiments in presence of Control factors and Noise factors.

1.2.2 Specific objectives

Two specific objectives are highlighted. The first specific objective is to present a methodology for analyzing the Double orthogonal array design and Combined array design using the criteria of Mean square error and Desirability function. The second specific objective is to propose a methodology for pairwise comparisons of treatments in Double orthogonal array design based on Signal-to-noise ratio.

1.3 Subdivisions of the thesis

The thesis is subdivided into four chapters. Chapter 1 concerns the general introduction. It presents the background information of robust designs and the objectives of the research. Chapter 2 presents a methodological approach for conducting robust design of experiments. The designs presented are the Double Orthogonal Array Design and Combined Array Design. The purpose of this chapter is to give a methodological approach of modelling both the mean and variance of a quality characteristic in robust designs. The data analysis is based upon the mean square error criterion and the desirability function. The chapter emphasizes the economic impact of conducting designed experiments in the industry. The impact is measured by applying the Quality Loss Function . This highlight the aspect of robustness proposed by Taguchi. Two examples borrowed from the existing literature on robust designs are presented. The first example is a robust design conducted on a chemical process.

1.3. Subdivisions of the thesis

The second example is a robust design conducted on elastic element of a force transducer. Chapter 3 describes a proposed method for conducting pairwise comparisons of treatments considering the metric known as Signal-to-noise ratio. Simulation studies and real examples are presented. Chapter 4 presents the general conclusions of this investigation.

Chapter 2

Robust Design of Experiments: A Methodological Approach

2.1 Double Orthogonal Array Design

The double orthogonal array design was initiated by Genichi Taguchi ([Taguchi and Wu, 1985](#)). It consists of a cross-product of two experimental designs. The first design, known as inner design, is a combination of the levels of the design factors. The second design, referred to as outer array design, is a combination of the levels of the noise factors. Each combination of the levels of the design factors forms an experiment. For each experiment, the same array of the noise factors is run.

Suppose that the quality characteristic, say y , of a product or a process depends on p design factors x_1, \dots, x_p and q noise factors z_1, \dots, z_q . Observations of the quality characteristic, say y_{ij} , are the combinations of the levels of the design factors ($i = 1, 2, \dots, n$) and the levels of the noise factors ($j = 1, 2, \dots, r$). The total number of runs required to conduct an experiment in this case is $n \times r$. The experimental structure of double orthogonal array design is represented by [Figure 2.1](#).

Data analysis consists of fitting a second order regression model of the form

$$y = \beta_0 + \mathbf{x}^T \boldsymbol{\beta} + \mathbf{x}^T \mathbf{B} \mathbf{x} + \epsilon. \quad (2.1)$$

In this model, \mathbf{x} is the vector of control factors, β_0 the intercept, $\boldsymbol{\beta}$ is a vector of coefficients of 1st order control factors, \mathbf{B} is a matrix of coefficients of 2nd order terms of control factors and their interactions, ϵ is a vector of random errors of the regression model. The random errors are assumed to be $N(0, \sigma^2)$. The response y is the mean or the variance of the quality characteristic.

Once the model in [Equation \(2.1\)](#) is adjusted to the data obtained from the experiment as presented in [Figure 2.1](#), the adjusted models for the mean and variance of the quality

2.2. Combined Array Design

Figure 2.1: Experimental structure of Double Orthogonal Array Design (DOAD).

	z_1	z_{11}	z_{21}	\dots	z_{r1}				
	z_2	z_{12}	z_{22}	\dots	z_{r2}				
	\vdots	\vdots	\vdots	\vdots	\vdots				
	z_q	z_{1q}	z_{2q}	\dots	z_{rq}				
x_1	x_2	\dots	x_p	Observations				\bar{y}	$\ln(s^2)$
x_{11}	x_{12}	\dots	x_{1p}	y_{11}	y_{12}	\dots	y_{1r}	\bar{y}_1	$\ln(s_1^2)$
x_{21}	x_{22}	\dots	x_{2p}	y_{21}	y_{22}	\dots	y_{2r}	\bar{y}_2	$\ln(s_2^2)$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
x_{n1}	x_{n2}	\dots	x_{np}	y_{n1}	y_{n2}	\dots	y_{nr}	\bar{y}_n	$\ln(s_n^2)$

characteristic, say \hat{y}_{Mean} and \hat{y}_{Var} , are obtained. Model parameters are estimated by the least squares method.

2.2 Combined Array Design

The combined array design is a single experimental design in control and noise factors. Both control and noise factors are then modelled. The results of the experiment can be described by a model with only a small number of main effects and low-order interactions. Significant design-by-noise interactions are interpreted as evidence of dispersion effects and are used to choose settings of design factors that minimize the process variation. The data obtained from the combined array design are analysed by fitting a model for the mean and variance.

Suppose that the quality characteristic y of a product or process depends on p design factors x_1, \dots, x_p and q noise factors z_1, \dots, z_q . The experimental structure of the combined array design is presented by Figure 2.2.

Figure 2.2: Experimental structure of Combined Array Design (CAD).

x_1	x_2	\dots	x_p	z_1	z_2	\dots	z_q	y
x_{11}	x_{12}	\dots	x_{1p}	z_{11}	z_{12}	\dots	z_{1q}	y_{11}
\vdots	\vdots	\vdots	\dots	\vdots	\vdots	\dots	\vdots	\vdots
x_{n1}	x_{n2}	\dots	x_{np}	z_{r1}	z_{r2}	\dots	z_{rq}	y_{nr}

2.2.1 Dual Response Surface Approach

Let the system be described by a variable $y(\mathbf{x}, \mathbf{z})$ that depends on a set of controllable factors (the vector \mathbf{x}) and a set of random noise factors (the vector \mathbf{z}).

2.2. Combined Array Design

To explore the dependence of y on \mathbf{x} and \mathbf{z} , the following model is assumed for the response, to accommodate control-by-noise interactions:

$$y(\mathbf{x}, \mathbf{z}) = \beta_0 + \boldsymbol{\beta}^T \mathbf{x} + \mathbf{x}^T \mathbf{B} \mathbf{x} + \boldsymbol{\gamma}^T \mathbf{z} + \mathbf{x}^T \boldsymbol{\Delta} \mathbf{z} + \epsilon. \quad (2.2)$$

In this model, \mathbf{z} is the random noise vector, ϵ 's are independent and identically distributed $N(0, \sigma^2)$ random errors. It is assumed that ϵ and \mathbf{z} are independent. The constant β_0 , the vectors $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ and the matrices \mathbf{B} and $\boldsymbol{\Delta}$ consist of unknown parameters, and σ^2 is also usually unknown. It is also assumed that $E(\mathbf{z}) = \mathbf{0}$ and that $Cov(\mathbf{z}) = \boldsymbol{\Omega}$ is known.

After the model (2.2) is fitted to the data from the designed experiment, the corresponding adjusted response model is given by the expression

$$\widehat{y(\mathbf{x}, \mathbf{z})} = \widehat{\beta}_0 + \mathbf{x}^T \widehat{\boldsymbol{\beta}} + \mathbf{x}^T \widehat{\mathbf{B}} \mathbf{x} + \widehat{\boldsymbol{\gamma}}^T \mathbf{z} + \mathbf{x}^T \widehat{\boldsymbol{\Delta}} \mathbf{z}. \quad (2.3)$$

The two response surfaces are obtained analytically from (2.3). Both response surfaces are in terms of control factors.

Result 2.1 Mean response surface and variance response surface

Consider the model given by Equation 2.3. Then the mean response surface, and the variance response surface are expressed as

$$E_z \left(\widehat{y(\mathbf{x}, \mathbf{z})} \right) = \widehat{\beta}_0 + \mathbf{x}^T \widehat{\boldsymbol{\beta}} + \mathbf{x}^T \widehat{\mathbf{B}} \mathbf{x} \quad \text{and} \quad (2.4)$$

$$Var_z \left(\widehat{y(\mathbf{x}, \mathbf{z})} \right) = \left(\widehat{\boldsymbol{\gamma}}^T + \mathbf{x}^T \widehat{\boldsymbol{\Delta}} \right) \boldsymbol{\Omega} \left(\widehat{\boldsymbol{\gamma}} + \widehat{\boldsymbol{\Delta}}^T \mathbf{x} \right), \quad (2.5)$$

respectively.

Proof:

Calculating the expected value and the variance of $\widehat{y(\mathbf{x}, \mathbf{z})}$ with respect to the random vector \mathbf{z} leads to

$$\begin{aligned} E_z \left(\widehat{y(\mathbf{x}, \mathbf{z})} \right) &= E_z \left[\widehat{\beta}_0 + \mathbf{x}^T \widehat{\boldsymbol{\beta}} + \mathbf{x}^T \widehat{\mathbf{B}} \mathbf{x} + \widehat{\boldsymbol{\gamma}}^T \mathbf{z} + \mathbf{x}^T \widehat{\boldsymbol{\Delta}} \mathbf{z} \right] \\ &= E_z \left[\widehat{\beta}_0 + \mathbf{x}^T \widehat{\boldsymbol{\beta}} + \mathbf{x}^T \widehat{\mathbf{B}} \mathbf{x} \right] + E_z \left[\widehat{\boldsymbol{\gamma}}^T \mathbf{z} + \mathbf{x}^T \widehat{\boldsymbol{\Delta}} \mathbf{z} \right] \\ &= E_z \left[\widehat{\beta}_0 + \mathbf{x}^T \widehat{\boldsymbol{\beta}} + \mathbf{x}^T \widehat{\mathbf{B}} \mathbf{x} \right] \\ &= \widehat{\beta}_0 + \mathbf{x}^T \widehat{\boldsymbol{\beta}} + \mathbf{x}^T \widehat{\mathbf{B}} \mathbf{x} \end{aligned}$$

2.2. Combined Array Design

and

$$\begin{aligned}
 \text{Var}_z \left(\widehat{y(\mathbf{x}, \mathbf{z})} \right) &= \text{Var}_z \left[\widehat{\beta}_0 + \mathbf{x}^T \widehat{\beta} + \mathbf{x}^T \widehat{\mathbf{B}} \mathbf{x} + \widehat{\gamma}^T \mathbf{z} + \mathbf{x}^T \widehat{\Delta} \mathbf{z} \right] \\
 &= \text{Var}_z \left[\widehat{\beta}_0 + \mathbf{x}^T \widehat{\beta} + \mathbf{x}^T \widehat{\mathbf{B}} \mathbf{x} \right] + \text{Var}_z \left[\widehat{\gamma}^T \mathbf{z} + \mathbf{x}^T \widehat{\Delta} \mathbf{z} \right] \\
 &= \text{Var}_z \left[\widehat{\gamma}^T \mathbf{z} + \mathbf{x}^T \widehat{\Delta} \mathbf{z} \right] \\
 &= \text{Var}_z \left[\left(\widehat{\gamma}^T + \mathbf{x}^T \widehat{\Delta} \right) \mathbf{z} \right] \\
 &= \left(\widehat{\gamma}^T + \mathbf{x}^T \widehat{\Delta} \right) \text{Var}(\mathbf{z}) \left(\widehat{\gamma}^T + \mathbf{x}^T \widehat{\Delta} \right)^T \\
 &= \left(\widehat{\gamma}^T + \mathbf{x}^T \widehat{\Delta} \right) \Omega \left(\widehat{\gamma} + \widehat{\Delta}^T \mathbf{x} \right) \quad \blacksquare
 \end{aligned}$$

In the literature, various methods of optimization have been developed in order to obtain the optimal solution for the mean of the quality characteristic while minimizing the variance of the process. [Mares and Domínguez \(2013\)](#) have summarized and compared those methods. [Myers and Carter \(1973\)](#), and [Myers and Vining \(1990\)](#) have introduced the method commonly used in the dual response surface approach. They first fit second order models to both primary and secondary response surfaces. In this case, they are respectively, the mean and variance. Then, they optimize the primary response subject to an appropriate constraint on the value of the secondary response.

The optimal solution for the mean response is obtained by solving the problem:

$$\begin{cases} \text{Optimize} & E_z \left(\widehat{y(\mathbf{x}, \mathbf{z})} \right) \\ \text{Subject to} & \text{Var}_z \left(\widehat{y(\mathbf{x}, \mathbf{z})} \right) = \sigma_0^2 \\ & \mathbf{x} \in R. \end{cases} \quad (2.6)$$

The optimal solution for the variance model is the solution of the following problem:

$$\begin{cases} \text{Minimize} & \text{Var}_z \left(\widehat{y(\mathbf{x}, \mathbf{z})} \right) \\ \text{Subject to} & E_z \left(\widehat{y(\mathbf{x}, \mathbf{z})} \right) = T \\ & \mathbf{x} \in R. \end{cases} \quad (2.7)$$

R is the experimental region.

[Lin and Tu \(1995\)](#) argue that this method of optimization may be misleading, because the variance, which is to be minimized in the process, is forced to a fixed value. They propose a new procedure based on the mean square error criterion.

2.3. Mean Square Error Approach

2.3 Mean Square Error Approach

The mean square error (MSE) is an effective criterion to combine the mean and the standard deviation responses in dual response optimization.

The MSE function for *the target is best* case is given by

$$\widehat{MSE} = \begin{cases} [\widehat{y}_{Mean} - T]^2 - \widehat{y}_{Var} & : \text{Case of DOAD} \\ \left[E_z \left(\widehat{y(\mathbf{x}, \mathbf{z})} \right) - T \right]^2 + Var_z \left(\widehat{y(\mathbf{x}, \mathbf{z})} \right) & : \text{Case of CAD} \end{cases} \quad (2.8)$$

where T is the target value.

The MSE function for *the smaller-the-better* case is determined as follows

$$\widehat{MSE} = \begin{cases} [\widehat{y}_{Mean}]^2 - \widehat{y}_{Var} & : \text{Case of DOAD} \\ \left[E_z \left(\widehat{y(\mathbf{x}, \mathbf{z})} \right) \right]^2 + Var_z \left(\widehat{y(\mathbf{x}, \mathbf{z})} \right) & : \text{Case of CAD} \end{cases} \quad (2.9)$$

The MSE function for *the larger-the-better* type is given by the expression

$$\widehat{MSE} = \begin{cases} [\widehat{y}_{Mean} - H]^2 - \widehat{y}_{Var} & : \text{Case of DOAD} \\ \left[E_z \left(\widehat{y(\mathbf{x}, \mathbf{z})} \right) - H \right]^2 + Var_z \left(\widehat{y(\mathbf{x}, \mathbf{z})} \right) & : \text{Case of CAD} \end{cases} \quad (2.10)$$

where H is the highest plausible value of \widehat{y}_{Mean} or $E_z \left(\widehat{y(\mathbf{x}, \mathbf{z})} \right)$.

The optimization problem to solve is then

$$\begin{cases} \text{Minimize} & \widehat{MSE} \\ \text{Subject to} & \mathbf{x} \in R. \end{cases} \quad (2.11)$$

The advantage of the MSE approach is that it does not require any constraints on the secondary response, and it can handle more realistic models and much more complicated models than polynomial models.

2.4 Desirability Function

The desirability function to simultaneously optimizing multiple equations was originally proposed by [Harrington \(1965\)](#). The common approach is to transform each response \widehat{y}_i into an individual function d_i that varies over the range $[0, 1]$.

2.4. Desirability Function

2.4.1 Individual desirability function

Depending on whether a particular response \widehat{y}_i is to be maximized, minimized, or assigned a target value, the corresponding desirability function, d_i , is defined as follows (Derringer and Suich (1980)):

- **The nominal-the-best (NTB) case:**

$$d_i = \begin{cases} 0 & \text{if } \widehat{y}_i(\mathbf{x}) \leq L_i \text{ or } \widehat{y}_i(\mathbf{x}) \geq U_i \\ \left(\frac{\widehat{y}_i(\mathbf{x}) - L_i}{T_i - L_i}\right)^r & \text{if } L_i < \widehat{y}_i(\mathbf{x}) < T_i \\ \left(\frac{\widehat{y}_i(\mathbf{x}) - U_i}{T_i - U_i}\right)^r & \text{if } T_i < \widehat{y}_i(\mathbf{x}) < U_i \\ 1 & \text{if } \widehat{y}_i(\mathbf{x}) = T_i \end{cases}$$

- **The smaller-the-better (STB) case:**

$$d_i = \begin{cases} 1 & \text{if } \widehat{y}_i(\mathbf{x}) \leq L_i \\ \left(\frac{\widehat{y}_i(\mathbf{x}) - U_i}{L_i - U_i}\right)^r & \text{if } L_i < \widehat{y}_i(\mathbf{x}) < U_i \\ 0 & \text{if } \widehat{y}_i(\mathbf{x}) \geq U_i \end{cases}$$

- **The larger-the-better (LTB) case:**

$$d_i = \begin{cases} 0 & \text{if } \widehat{y}_i(\mathbf{x}) \leq L_i \\ \left(\frac{\widehat{y}_i(\mathbf{x}) - L_i}{U_i - L_i}\right)^r & \text{if } L_i < \widehat{y}_i(\mathbf{x}) < U_i \\ 1 & \text{if } \widehat{y}_i(\mathbf{x}) \geq U_i \end{cases}$$

The values of L_i and U_i are some acceptable lower bound and upper bound for $\widehat{y}_i(\mathbf{x})$, and T_i is the target value. The value of r can be chosen so that the desirability criterion is easier or more difficult to satisfy, indicating the weight of $\widehat{y}_i(\mathbf{x})$ in the process.

2.4.2 Overall desirability function

For an n responses system, the overall performance of the system is determined by the overall desirability D , which can be expressed as the geometric mean:

$$D = \left(\prod_{i=1}^n d_i \right)^{1/n}. \quad (2.12)$$

The optimization problem to solve is then

$$\begin{cases} \text{Maximize} & D \\ \text{Subject to} & \mathbf{x} \in R. \end{cases} \quad (2.13)$$

2.5. Desirability Function Based on Mean Square Error Criterion

2.5 Desirability Function Based on Mean Square Error Criterion

Suppose that k quality characteristics $(y^{(1)}, y^{(2)}, \dots, y^{(k)})$ of a product or process depend on p design factors (x_1, x_2, \dots, x_p) and q noise factors (z_1, z_2, \dots, z_q) . The experimental structure of the combined array design in the case of multiple responses is presented by Figure 2.3.

Figure 2.3: Experimental structure of Combined array design with multiple responses.

x_1	\dots	x_p	z_1	\dots	z_q	$y^{(1)}$	\dots	$y^{(k)}$
x_{11}	\dots	x_{1p}	z_{11}	\dots	z_{1q}	$y_{11}^{(1)}$	\dots	$y_{11}^{(k)}$
\vdots	\dots	\vdots	\vdots	\dots	\vdots	\vdots	\vdots	\vdots
x_{n1}	\dots	x_{np}	z_{r1}	\dots	z_{rq}	$y_{nr}^{(1)}$	\dots	$y_{nr}^{(k)}$

We consider the quadratic model described by Myers et al. (1992). For the i^{th} response, the matrix notation of the model can be written as follows:

$$y^{(i)}(\mathbf{x}, \mathbf{z}) = \beta_0^{(i)} + \boldsymbol{\beta}^{(i)T} \mathbf{x} + \mathbf{x}^T \mathbf{B}^{(i)} \mathbf{x} + \boldsymbol{\gamma}^{(i)T} \mathbf{z} + \mathbf{x}^T \boldsymbol{\Delta}^{(i)} \mathbf{z} + \epsilon^{(i)}; i = 1, \dots, k, \quad (2.14)$$

where

- $y^{(i)}(\mathbf{x}, \mathbf{z})$ denotes the i^{th} response,
- \mathbf{x} denotes the vector of control factors,
- \mathbf{z} denotes the vector of noise factors,
- $\beta_0^{(i)}$ is the intercept,
- $\boldsymbol{\beta}^{(i)}$ is the vector of coefficients for the linear effects in control variables,
- $\mathbf{B}^{(i)}$ is the matrix whose main diagonals are the regression coefficients associated with the pure quadratic effects of the control factors and whose off-diagonals are one-half of the mixed quadratic (interaction) effects of the control factors,
- $\boldsymbol{\gamma}^{(i)}$ is the vector of coefficients for the linear effects in the noise factors,
- $\boldsymbol{\Delta}^{(i)}$ is the matrix of the control-by-noise interaction coefficients,
- $\epsilon^{(i)}$ is a random error.

It is assumed that the $\epsilon^{(i)}$ s are independently and identically distributed $N(0, \sigma_{\epsilon^{(i)}}^2)$ and that all noise factors are continuous. It is also assumed that, in accordance with design level centering and scaling, $E(\mathbf{z}) = \mathbf{0}$ and $Var(\mathbf{z}) = \boldsymbol{\Omega} = \sigma_z^2 \mathbf{I}$.

2.5. Desirability Function Based on Mean Square Error Criterion

Considering these assumptions and using expectation and variance operations taken over \mathbf{z} on Equation (2.14), we derive the mean and variance response surfaces.

Result 2.2 *Mean response surface and variance response surface for the i^{th} response variable*

Consider the model given by Equation (2.14). Then the mean response surface, and the variance response surface corresponding to the i^{th} response variable are

$$\begin{aligned} E_{\mathbf{z}} (y^{(i)}(\mathbf{x}, \mathbf{z})) &= \beta_0^{(i)} + \boldsymbol{\beta}^{(i)T} \mathbf{x} + \mathbf{x}^T \mathbf{B}^{(i)} \mathbf{x}; i = 1, \dots, k. \text{ and} \\ \text{Var}_{\mathbf{z}} (y^{(i)}(\mathbf{x}, \mathbf{z})) &= \left(\boldsymbol{\gamma}^{(i)T} + \mathbf{x}^T \boldsymbol{\Delta}^{(i)} \right) \Omega \left(\boldsymbol{\gamma}^{(i)} + \boldsymbol{\Delta}^{(i)T} \mathbf{x} \right) + \sigma_{\epsilon^{(i)}}^2; i = 1, \dots, k, \end{aligned} \quad (2.15)$$

respectively.

Proof:

Calculating the expected value and the variance of $y^{(i)}(\mathbf{x}, \mathbf{z})$ with respect to the random vector \mathbf{z} leads to

$$\begin{aligned} E_{\mathbf{z}} (y^{(i)}(\mathbf{x}, \mathbf{z})) &= E_{\mathbf{z}} \left[\beta_0^{(i)} + \boldsymbol{\beta}^{(i)T} \mathbf{x} + \mathbf{x}^T \mathbf{B}^{(i)} \mathbf{x} + \boldsymbol{\gamma}^{(i)T} \mathbf{z} + \mathbf{x}^T \boldsymbol{\Delta}^{(i)} \mathbf{z} + \epsilon^{(i)} \right] \\ &= E_{\mathbf{z}} \left[\beta_0^{(i)} + \boldsymbol{\beta}^{(i)T} \mathbf{x} + \mathbf{x}^T \mathbf{B}^{(i)} \mathbf{x} \right] + E_{\mathbf{z}} \left[\boldsymbol{\gamma}^{(i)T} \mathbf{z} + \mathbf{x}^T \boldsymbol{\Delta}^{(i)} \mathbf{z} + \epsilon^{(i)} \right] \\ &= E_{\mathbf{z}} \left[\beta_0^{(i)} + \boldsymbol{\beta}^{(i)T} \mathbf{x} + \mathbf{x}^T \mathbf{B}^{(i)} \mathbf{x} \right] \\ &= \beta_0^{(i)} + \boldsymbol{\beta}^{(i)T} \mathbf{x} + \mathbf{x}^T \mathbf{B}^{(i)} \mathbf{x} \end{aligned}$$

and

$$\begin{aligned} \text{Var}_{\mathbf{z}} (y^{(i)}(\mathbf{x}, \mathbf{z})) &= \text{Var}_{\mathbf{z}} \left[\beta_0^{(i)} + \boldsymbol{\beta}^{(i)T} \mathbf{x} + \mathbf{x}^T \mathbf{B}^{(i)} \mathbf{x} + \boldsymbol{\gamma}^{(i)T} \mathbf{z} + \mathbf{x}^T \boldsymbol{\Delta}^{(i)} \mathbf{z} + \epsilon^{(i)} \right] \\ &= \text{Var}_{\mathbf{z}} \left[\beta_0^{(i)} + \boldsymbol{\beta}^{(i)T} \mathbf{x} + \mathbf{x}^T \mathbf{B}^{(i)} \mathbf{x} \right] + \text{Var}_{\mathbf{z}} \left[\boldsymbol{\gamma}^{(i)T} \mathbf{z} + \mathbf{x}^T \boldsymbol{\Delta}^{(i)} \mathbf{z} + \epsilon^{(i)} \right] \\ &= \text{Var}_{\mathbf{z}} \left[\boldsymbol{\gamma}^{(i)T} \mathbf{z} + \mathbf{x}^T \boldsymbol{\Delta}^{(i)} \mathbf{z} + \epsilon^{(i)} \right] \\ &= \text{Var}_{\mathbf{z}} \left[\left(\boldsymbol{\gamma}^{(i)T} + \mathbf{x}^T \boldsymbol{\Delta}^{(i)} \right) \mathbf{z} + \epsilon^{(i)} \right] \\ &= \left(\boldsymbol{\gamma}^{(i)T} + \mathbf{x}^T \boldsymbol{\Delta}^{(i)} \right) \text{Var}(\mathbf{z}) \left(\boldsymbol{\gamma}^{(i)T} + \mathbf{x}^T \boldsymbol{\Delta}^{(i)} \right)^T + \text{Var}_{\mathbf{z}} \left[\epsilon^{(i)} \right] \\ &= \left(\boldsymbol{\gamma}^{(i)T} + \mathbf{x}^T \boldsymbol{\Delta}^{(i)} \right) \Omega \left(\boldsymbol{\gamma}^{(i)} + \boldsymbol{\Delta}^{(i)T} \mathbf{x} \right) + \sigma_{\epsilon^{(i)}}^2; i = 1, \dots, k \quad \blacksquare \end{aligned}$$

After fitting the models in Equation (2.15) to the data, the corresponding adjusted response models are given by the expressions:

$$\begin{aligned} \widehat{E}_{\mathbf{z}} (y^{(i)}(\mathbf{x}, \mathbf{z})) &= \widehat{\beta}_0^{(i)} + \widehat{\boldsymbol{\beta}}^{(i)T} \mathbf{x} + \mathbf{x}^T \widehat{\mathbf{B}}^{(i)} \mathbf{x}; i = 1, \dots, k. \\ \widehat{\text{Var}}_{\mathbf{z}} (y^{(i)}(\mathbf{x}, \mathbf{z})) &= \left(\widehat{\boldsymbol{\gamma}}^{(i)T} + \mathbf{x}^T \widehat{\boldsymbol{\Delta}}^{(i)} \right) \Omega \left(\widehat{\boldsymbol{\gamma}}^{(i)} + \widehat{\boldsymbol{\Delta}}^{(i)T} \mathbf{x} \right) + \widehat{\sigma}_{\epsilon^{(i)}}^2; i = 1, \dots, k. \end{aligned} \quad (2.16)$$

2.5. Desirability Function Based on Mean Square Error Criterion

The values of error variances $\widehat{\sigma}_{\epsilon^{(i)}}^2; i = 1, \dots, k$ are the residual mean squares obtained from fitting the model to the data.

As we aim to apply the desirability function approach based on the mean square error criterion, we determine the individual mean square error function for each response, i.e., $\widehat{MSE}^{(i)}; i = 1, \dots, k$, depending upon the purposes of the experiment. We calculate the corresponding individual desirability function:

$$d^{(i)} = h\left(\widehat{MSE}^{(i)}(\mathbf{x})\right); i = 1, \dots, k. \quad (2.17)$$

We now determine the individual desirability functions.

2.5.1 Individual desirability functions based on mean square error criterion

For the smaller-the-better type, the individual desirability function of the i^{th} mean square error is given by

$$d^{(i)} = \begin{cases} 1 & \text{if } \widehat{MSE}^{(i)}(\mathbf{x}) \leq L^{(i)} \\ \left(\frac{\widehat{MSE}^{(i)}(\mathbf{x}) - U^{(i)}}{L^{(i)} - U^{(i)}}\right)^r & \text{if } L^{(i)} < \widehat{MSE}^{(i)}(\mathbf{x}) < U^{(i)} \\ 0 & \text{if } \widehat{MSE}^{(i)}(\mathbf{x}) \geq U_i^{(i)}; i = 1, \dots, k \end{cases} \quad (2.18)$$

where $L^{(i)} = \widehat{MSE}^{(i)}(\mathbf{x}_{\min})$ and $U^{(i)} = \widehat{MSE}^{(i)}(\mathbf{x}_{\max})$. \mathbf{x}_{\min} and \mathbf{x}_{\max} are respectively solutions of the following optimization problems:

$$\begin{cases} \text{Minimize } \widehat{MSE}^{(i)}(\mathbf{x}) \\ \text{Subject to } \mathbf{x} \in R; i = 1, \dots, k \end{cases} \quad \text{and} \quad \begin{cases} \text{Maximize } \widehat{MSE}^{(i)}(\mathbf{x}) \\ \text{Subject to } \mathbf{x} \in R; i = 1, \dots, k \end{cases}$$

The overall desirability function is determined as

$$D = \left(\prod_{i=1}^k d^{(i)}\right)^{1/k}. \quad (2.19)$$

The common optimal setting for k response variables is obtained by solving the following optimization problem:

$$\begin{cases} \text{Maximize } D \\ \text{Subject to } \mathbf{x} \in \mathbf{R}. \end{cases} \quad (2.20)$$

2.6. Economic Impact of Designed Experiments

2.6 Economic Impact of Designed Experiments

In this section, we assess the economic impact of designed experiment using the quadratic loss function. The quadratic loss function (QLF) is a metric used to provide a better estimate of the monetary loss incurred by manufacturers and consumers when the product performance deviates from its target value [Meng et al. \(2010\)](#). Then, the QLF is used to evaluate the economic impact of conducting an experiment on the process.

The QLF is given by the expression

$$L(y) = k(y - T)^2, \quad (2.21)$$

where y is the quality characteristic of a product or process, T is the target value and k is the quality loss coefficient.

The expected quality loss (EQL) is

$$Q = E[L(y)] = kE(y - T)^2 = k[(E(y) - T)^2 + Var(y)].$$

By taking $E(y) = \mu$ and $Var(y) = \sigma^2$, the expected quality loss becomes

$$Q = k[(\mu - T)^2 + \sigma^2]. \quad (2.22)$$

Then the estimate of expected quality loss is:

$$\hat{Q} = k[(\hat{\mu} - T)^2 + \hat{\sigma}^2], \quad (2.23)$$

where $\hat{\mu} = \bar{y}$ and $\hat{\sigma}^2 = s^2$.

The quality loss coefficient k is determined by first finding the functional limits or customer tolerance for y . The function limits are determined by $T \pm \Delta_0$. These are the points at which the product would fail or produce unacceptable performance in approximately half of the customer applications. Let A_0 be the value of the quality loss function at $T \pm \Delta_0$, that is $L(y) = A_0$ at $y = T \pm \Delta_0$. Substituting the functional limits $T \pm \Delta_0$ and the value of the quality loss into Equation (2.21), the quality loss coefficients is found to be

$$k = \frac{A_0}{(\Delta_0)^2}.$$

2.6.1 Types of Quadratic Loss Functions

While conducting an experiment, the designer is interested in reaching the target value or minimizing or maximizing the value of the quality characteristic. These three cases of quality characteristics are referred to as *the nominal-the-best type* (NTB), *the smaller-the-better* (STB) and *the larger-the-better* (LTB), respectively.

2.7. Illustrative Examples

Let $y^T = (y_1, y_2, \dots, y_n)$ be a vector of observations where y is the quality characteristic of a product or process. Table 2.1 shows the types of QLF and the average quadratic loss functions corresponding to each kind of quality characteristic of interest. More details on this section can be found in [Fowlkes and Creveling \(1995\)](#).

Table 2.1: Types of Quadratic Loss Functions.

Type	Quadratic Loss	Expected Quality Loss	Estimate of Expected Quality Loss
NTB	$L(y) = k(y - T)^2$	$Q = k((\mu - T)^2 + \sigma^2)$	$\widehat{Q} = k((\bar{y} - T)^2 + s^2)$
STB	$L(y) = ky^2$	$Q = k(\mu^2 + \sigma^2)$	$\widehat{Q} = k(\bar{y}^2 + s^2)$
LTB	$L(y) = \frac{k'}{y^2}$	$Q = \frac{k'}{\mu^2} \left(1 + \frac{3\sigma^2}{\mu^2}\right)$	$\widehat{Q} = \frac{k'}{\bar{y}^2} \left(1 + \frac{3s^2}{\bar{y}^2}\right)$

In this table, $k = \frac{A_0}{(\Delta_0)^2}$ and $k' = A_0 (\Delta_0)^2$.

2.7 Illustrative Examples

2.7.1 Example 1:

Robust design conducted on a chemical process

Problem statement and Design

The application is a chemical process adapted from ([Lawson, 2010](#)). In this process, side reactions create tars that result in lower product quality. When the level of tars produced is too high, yield decreases and further blending must be done with the finished product to decrease tar to a level that is acceptable to customers. The tars were thought to be created by reactions involving impurities in the most expensive reagent, A, and other impurities that accumulate in the recycle solvent stream. The proposed method to solving the problem was to experiment with the setting of process variables, or design parameters, to see if operating conditions could be found that were less sensitive to impurities in reagent A and the solvent stream ([Lawson, 1990](#)).

The objectives of conducting a designed experiment are to diminish the proportion of impurities and to reduce the variance of the process.

The response variable is the proportion of impurities (in percentage). The factors involved in this experiment are 3 design factors and 2 noise factors. The design factors are x_1 : reaction temperature, x_2 : the catalyst concentration, x_3 : the excess of reagent B. The noise factors are z_1 : purity of reagent A, z_2 : purity of the solvent stream. It is assumed that z_1 and z_2 are uncorrelated and $\sigma_{z_1}^2 = \sigma_{z_2}^2 = 1$ so that the variance-covariance matrix, say $cov(z) = \Omega$,

2.7. Illustrative Examples

is an identity matrix of dimension 2.

Experiments are performed at combinations of levels of the design factors defined by a Box-Behnken design. The combinations of levels of the noise factors are arranged in a 2^2 factorial design. This means that an experiment for 15 combinations of the control factors is realised, and each of these is repeated in each of the possible combinations of the noise factors. Coded and real levels of the factors, and the results obtained from the experiment are shown in Table A.1.

Methods of analysis

As the aim of the experiment is to diminish the proportion of impurities, it is convenient to apply the optimization method proposed by Lin and Tu (1995). In Harrington (1965), the method is referred to as Mean Square Error Criterion. In this application, the target $T = 0$.

In addition, we apply the desirability function to obtain the optimal settings. For convenience, full models are considered, regardless of the presence of some nonsignificance effects in the fitted models.

Results

The estimated mean response and estimated variance response surfaces are:

$$\begin{aligned}\hat{y}_{\text{Mean}} &= 14.80 - 8.17x_1 - 9.09x_2 - 0.14x_3 + 0.52x_1^2 + 8.30x_1x_2 + 0.07x_1x_3 + 5.01x_2^2 \\ &\quad + 0.18x_2x_3 + 0.18x_3^2. \\ \hat{y}_{\text{Var}} &= \exp(2.59 + 0.03x_1 - 1.66x_2 + 0.58x_3 - 0.21x_1^2 + 0.03x_1x_2 + 0.02x_1x_3 + 0.34x_2^2 \\ &\quad - 0.07x_2x_3 + 0.48x_3^2).\end{aligned}$$

- **MSE approach**

The optimization problem to solve is:

$$\left\{ \begin{array}{l} \text{Minimize} \quad (14.80 - 8.17x_1 - 9.09x_2 - 0.14x_3 + 0.52x_1^2 + 8.30x_1x_2 + 0.07x_1x_3 \\ \quad + 5.01x_2^2 + 0.18x_2x_3 + 0.18x_3^2)^2 + \exp(2.59 + 0.03x_1 - 1.66x_2 \\ \quad + 0.58x_3 - 0.21x_1^2 + 0.03x_1x_2 + 0.02x_1x_3 + 0.34x_2^2 - 0.07x_2x_3 + 0.48x_3^2) \\ \text{Subject to} \quad -1 \leq x_1 \leq 1 \\ \quad \quad \quad -1 \leq x_2 \leq 1 \\ \quad \quad \quad -1 \leq x_3 \leq 1. \end{array} \right.$$

The solution of this optimization problem uses the *nloptr* package (Ypma (2015)). Contour and perspective plots of the mean, variance and MSE responses (in Appendix C.1) are sketched using the *rsm* package (Lenth (2015)).

2.7. Illustrative Examples

- *Desirability function approach*

The lower and upper bound for the mean response are respectively given by $L_{Mean} = \hat{y}_{Mean}(\mathbf{x}_{min})$ and $U_{Mean} = \hat{y}_{Mean}(\mathbf{x}_{max})$, where \mathbf{x}_{min} and \mathbf{x}_{max} are respectively solutions of the following optimization problems:

$$\left\{ \begin{array}{l} \text{Minimize} \quad \hat{y}_{Mean} \\ \text{Subject to} \quad \mathbf{x} \in R \end{array} \right. \text{ and } \left\{ \begin{array}{l} \text{Maximize} \quad \hat{y}_{Mean} \\ \text{Subject to} \quad \mathbf{x} \in R. \end{array} \right.$$

Similar logic leads to $L_{Var} = \hat{y}_{Var}(\mathbf{x}_{min})$ and $U_{Var} = \hat{y}_{Var}(\mathbf{x}_{max})$, where \mathbf{x}_{min} and \mathbf{x}_{max} are respectively solutions of the optimization problems:

$$\left\{ \begin{array}{l} \text{Minimize} \quad \hat{y}_{Var} \\ \text{Subject to} \quad \mathbf{x} \in R \end{array} \right. \text{ and } \left\{ \begin{array}{l} \text{Maximize} \quad \hat{y}_{Var} \\ \text{Subject to} \quad \mathbf{x} \in R. \end{array} \right.$$

Table 2.2 displays the values obtained.

Table 2.2: Lower bounds and upper bounds of individual desirability functions.

$L_{Mean} : 7.11$	$L_{Var} : 2.21$
$U_{Mean} : 46.5$	$U_{Var} : 79.04$

The individual desirability functions for the proportion of impurities and its variance are respectively

$$d(\hat{y}_{Mean}) = \begin{cases} 1 & \text{if } \hat{y}_{Mean} \leq 7.11 \\ \frac{\hat{y}_{Mean}-46.5}{7.11-46.5} & \text{if } 7.11 < \hat{y}_{Mean} < 46.5 \\ 0 & \text{if } \hat{y}_{Mean} \geq 46.5 \end{cases}$$

and

$$d(\hat{y}_{Var}) = \begin{cases} 1 & \text{if } \hat{y}_{Var} \leq 2.21 \\ \frac{\hat{y}_{Var}-79.04}{2.21-79.04} & \text{if } 2.21 < \hat{y}_{Var} < 79.04 \\ 0 & \text{if } \hat{y}_{Var} \geq 79.04. \end{cases}$$

The overall desirability function is

$$D = \sqrt{d(\hat{y}_{Mean}) \times d(\hat{y}_{Var})}.$$

2.7. Illustrative Examples

The optimization problem to solve is the following:

$$\begin{cases} \text{Maximize} & D \\ \text{Subject to} & -1 \leq x_1 \leq 1 \\ & -1 \leq x_2 \leq 1 \\ & -1 \leq x_3 \leq 1. \end{cases}$$

We adapt the *desirability* package (Kuhn (2015)) to solve the previous optimization problem. The results of Example 1 are summarized in Table 2.3.

Table 2.3: Optimal conditions for the chemical process problem.

Approach	Statistic	Optimal results
Mean Square Error	x_{Opt}	(1,0.16405,-0.32174)
	$\hat{y}_{\text{Mean}}(x_{\text{Opt}})$	7.19
	$\hat{y}_{\text{Var}}(x_{\text{Opt}})$	7.42
	$\overline{\text{MSE}}$	59.1
Desirability Function	x_{Opt}	(1, 0.43189, -0.57353)
	$\hat{y}_{\text{Mean}}(x_{\text{Opt}})$	7.8
	$\hat{y}_{\text{Var}}(x_{\text{Opt}})$	4.79
	$\overline{\text{MSE}}$	65.6
	Desirability	0.97446

As far as it may concern, if the main objective of the experiment is to reduce the proportion of impurities, the mean square error approach gives the best results. On the contrary, if the interest is the reduction of variability in the process, the desirability function seems to be the best method of analysis.

When both the reduction of proportion of impurities and reduction of variability are of utmost importance, the selection of the design to be conducted and the method of analysis to be adopted will take into account the estimated value of the corresponding quality loss function.

The quality loss function accounts for the loss in terms monetary funds incurred by the deviation from the target and high variability in the process. Example 2 illustrates the use of quality loss function.

2.7.2 Example 2: Economic impact of a robust design conducted on a chemical process

The process is working in real conditions of $x_1 = 200$, $x_2 = 28$ and $x_3 = 17$ for the reaction temperature, catalyst concentration and excess of reagent B, respectively. A sample of 24 data is taken to verify the conditions of impurities. These refer to the mean and variance

2.7. Illustrative Examples

of the process. The sample values are: 23.08, 23.01, 18.11, 20.14, 47.14, 23.20, 17.12, 18.93, 21.42, 22.72, 20.36, 24.99, 32.77, 25.26, 19.50, 23.14, 23.09, 30.49, 35.72, 21.95, 26.43, 24.48, 34.21, 25.03 (Mares & Domínguez, 2013).

The corresponding sample mean and sample variance are, respectively

$$\bar{y} = \frac{1}{24} \sum_{i=1}^{24} y_i = 25.1 \text{ and } s^2 = \frac{1}{23} \sum_{i=1}^{24} (y_i - \bar{y})^2 = 45.64.$$

Therefore, the estimate of the expected quality loss (EQL) for the quadratic loss is:

$$\widehat{Q}_{process} = k(\bar{y}^2 + s^2) = (25.1)^2 + 45.64 = 675.65,$$

where $k = 1$. This constant in practice plays a very important role because it indicates the overall cost of poor quality.

We now calculate the estimate of the expected quality loss of the optimal process conditions as given by Table 2.1.

The estimate of the expected quality loss of the designed experiment is defined as

$$\widehat{Q}_{doe} = \widehat{y}_{\text{Mean}}(\mathbf{x}_{\text{Opt}})^2 + \widehat{y}_{\text{Var}}(\mathbf{x}_{\text{Opt}})^2.$$

The overall difference between the estimate of the expected quality loss of the real process and the estimate of the expected quality loss of the optimal process conditions of the designed experiment, this is $\Delta = \widehat{Q}_{process} - \widehat{Q}_{doe}$, indicates the expected gain incurred by conducting the process in its optimal conditions. Table 2.4 presents the results.

Table 2.4: Economic impact of designed experiment.

Approach	Economic impact	Double Orthogonal Array Design
Mean Square Error	\widehat{Q}_{doe}	59.116
	Δ	616.53
Desirability Function	\widehat{Q}_{doe}	65.63
	Δ	610.02

We can see that the estimate of the expected quality loss of the designed experiment, \widehat{Q}_{doe} , ranges between 59.116 and 65.63 monetary units, while the overall difference that indicates the gain incurred by conducting the process under the optimal conditions, Δ , ranges between 610.02 and 616.53 monetary units.

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2.7.3 Example 3: Robust design conducted on the elastic element of a force transducer

Problem statement and Design

The application is a robust design conducted on the elastic element of a force transducer. This is a case study presented by Romano et al. (2004) and revisited by Köksoy (2006). A transducer is a device that provides an output quantity having a determined relationship to the parameter being measured, the force in this case.

When a compressive load is applied to the elastic element a peculiar strain pattern is created over the central section of the elastic element, where strain peaks due to design factors. The deformation of the element is then measured by a second device which converts it into a measurable output.

The design of the element is intended to minimize the transducer's inaccuracy, which originates from two major sources, namely non-linearity and hysteresis. These two indicators define responses y_1 and y_2 . The non-linearity effect is the ratio between longitudinal strain and transversal strain. The hysteresis indicator is the ratio between maximum strain on the measuring area and longitudinal strain.

This example involves a combined array design with three control factors (\mathbf{x}) and two noise factors (\mathbf{z}). Control factors are the three parameters defining the element configuration, namely lozenge angle (x_1), bore diameter (x_2), and half-length of the vertical segment (x_3). Noise factors are the deviation of the lozenge angle from its nominal value z_1 and the deviation of the bore diameter from its nominal value z_2 . These internal noise factors are admittedly independent. They are assumed to be normally distributed with zero mean and variances of σ_1^2 and σ_2^2 . Table A.2 in Appendices displays the coded and real levels of the factors, and the results obtained from the experiment.

The aim of this experiment is to find the settings for the lozenge angle (x_1), bore diameter (x_2), and half-length of the vertical segment (x_3) which minimize the hysteresis (y_2) and achieve a target value of 1 for the non-linearity indicator (y_1).

The experimental design used is a central composite design (CCD) made up of a half-fraction design of a five factor-two level factorial design, star points for control factors only, and a threefold center point.

Methods of analysis

As the problem is a multiple response case, we apply the desirability function based on mean square error criterion. It is assumed that z_1 and z_2 are uncorrelated and $\sigma_{z_1}^2 = \sigma_{z_2}^2 = 1$.

Hence, $\Omega = Var(\mathbf{z}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

2.7. Illustrative Examples

Results

The following prediction equations are obtained for non-linearity and hysteresis indicator:

$$\begin{aligned} \widehat{y^{(1)}}(\mathbf{x}, \mathbf{z}) &= 1.377 - 0.361x_1 - 0.155x_2 + 0.077x_3 + 0.042x_1^2 + 0.007x_2^2 + 0.002x_3^2 - 0.148x_1x_2 \\ &\quad + 0.022x_1x_3 + 0.013x_2x_3 - 0.059z_1 - 0.012z_2 + 0.010x_1z_1 - 0.008x_1z_2 - 0.006x_2z_1 \\ &\quad + 0.001x_2z_2 + 0.005x_3z_1 + 0.003x_3z_2. \end{aligned}$$

$$\begin{aligned} \widehat{y^{(2)}}(\mathbf{x}, \mathbf{z}) &= 1.660 + 0.592x_1 + 0.438x_2 - 0.095x_3 + 0.247x_1^2 - 0.123x_2^2 + 0.047x_3^2 + 0.301x_1x_2 \\ &\quad - 0.143x_1x_3 - 0.033x_2x_3 + 0.066z_1 - 0.042z_2 + 0.079x_1z_1 + 0.017x_1z_2 - 0.031x_2z_1 \\ &\quad - 0.061x_2z_2 - 0.004x_3z_1 - 0.014x_3z_2. \end{aligned}$$

The corresponding estimated mean models are:

$$\begin{aligned} \widehat{E}_{\mathbf{z}}(y^{(1)}(\mathbf{x}, \mathbf{z})) &= 1.377 - 0.361x_1 - 0.155x_2 + 0.077x_3 + 0.042x_1^2 + 0.007x_2^2 + 0.002x_3^2 \\ &\quad - 0.148x_1x_2 + 0.022x_1x_3 + 0.013x_2x_3. \\ \widehat{E}_{\mathbf{z}}(y^{(2)}(\mathbf{x}, \mathbf{z})) &= 1.660 + 0.592x_1 + 0.438x_2 - 0.095x_3 + 0.247x_1^2 - 0.123x_2^2 + 0.047x_3^2 \\ &\quad + 0.301x_1x_2 - 0.143x_1x_3 - 0.033x_2x_3. \end{aligned}$$

Details on related calculations are given in Appendix B.1.

The models for the variances are given by:

$$\begin{aligned} \widehat{Var}_{\mathbf{z}}(y^{(1)}(\mathbf{x}, \mathbf{z})) &= 3.925 \times 10^{-3} - 9.88 \times 10^{-4}x_1 + 6.84 \times 10^{-4}x_2 - 6.62 \times 10^{-4}x_3 \\ &\quad + 1.64 \times 10^{-4}x_1^2 + 3.7 \times 10^{-5}x_2^2 + 3.4 \times 10^{-5}x_3^2 - 1.36 \times 10^{-4}x_1x_2 \\ &\quad + 5.2 \times 10^{-5}x_1x_3 - 5.4 \times 10^{-5}x_2x_3. \\ \widehat{Var}_{\mathbf{z}}(y^{(2)}(\mathbf{x}, \mathbf{z})) &= 0.04312 + 0.009x_1 + 1.032 \times 10^{-3}x_2 + 6.48 \times 10^{-4}x_3 + 0.00653x_1^2 \\ &\quad + 4.682 \times 10^{-3}x_2^2 + 2.12 \times 10^{-4}x_3^2 - 6.972 \times 10^{-3}x_1x_2 \\ &\quad - 1.108 \times 10^{-4}x_1x_3 + 1.956x_2x_3. \end{aligned}$$

Detailed calculations are displayed in Appendix B.2.

Next, the individual MSE functions are computed from the two responses, i.e., $\widehat{MSE}^{(1)}$ for the non-linearity indicator and $\widehat{MSE}^{(2)}$ for the hysteresis indicator. These functions are given by the following expressions:

$$\begin{aligned} \widehat{MSE}_{NTB}^{(1)} &= \left[\widehat{E}_{\mathbf{z}}(y^{(1)}(\mathbf{x}, \mathbf{z})) - 1 \right]^2 + \widehat{Var}_{\mathbf{z}}(y^{(1)}(\mathbf{x}, \mathbf{z})). \\ \widehat{MSE}_{STB}^{(2)} &= \left[\widehat{E}_{\mathbf{z}}(y^{(2)}(\mathbf{x}, \mathbf{z})) \right]^2 + \widehat{Var}_{\mathbf{z}}(y^{(2)}(\mathbf{x}, \mathbf{z})). \end{aligned}$$

As the MSE criterion implies the minimization of the MSE, the corresponding desirability

2.7. Illustrative Examples

function is of the smaller-the-better type. The individual desirability function corresponding to each of the individual mean square error, i.e., $MSE^{(1)}$ and $MSE^{(2)}$ is given by:

$$d^{(i)} = h\left(\widehat{MSE}^{(i)}(\mathbf{x})\right) = \begin{cases} 1 & \text{if } \widehat{MSE}^{(i)}(\mathbf{x}) \leq L^{(i)} \\ \left(\frac{\widehat{MSE}^{(i)}(\mathbf{x}) - U^{(i)}}{L^{(i)} - U^{(i)}}\right)^r & \text{if } L^{(i)} < \widehat{MSE}^{(i)}(\mathbf{x}) < U^{(i)} \\ 0 & \text{if } \widehat{MSE}^{(i)}(\mathbf{x}) \geq U^{(i)}; i = 1, 2; r = 1 \end{cases}$$

where $L^{(i)} = \widehat{MSE}^{(i)}(\mathbf{x}_{\min})$ and $U^{(i)} = \widehat{MSE}^{(i)}(\mathbf{x}_{\max})$. \mathbf{x}_{\min} and \mathbf{x}_{\max} are respectively solutions of the following optimization problems:

$$\begin{cases} \text{Minimize} & \widehat{MSE}^{(i)}(\mathbf{x}) \\ \text{Subject to} & \mathbf{x} \in R; i = 1, 2 \end{cases} \text{ and } \begin{cases} \text{Maximize} & \widehat{MSE}^{(i)}(\mathbf{x}) \\ \text{Subject to} & \mathbf{x} \in R; i = 1, 2. \end{cases}$$

Table 2.5 displays the values for $L^{(i)}$'s and $U^{(i)}$'s.

Table 2.5: Ranges for individual desirability functions.

$L^{(1)} :$	0.00281	$L^{(2)} :$	1.06
$U^{(1)} :$	0.07277	$U^{(2)} :$	8.42

Hence, individual desirability functions are, respectively

$$d^{(1)} = h\left(\widehat{MSE}^{(1)}(\mathbf{x})\right) = \begin{cases} 1 & \text{if } \widehat{MSE}^{(1)}(\mathbf{x}) \leq 0.00281 \\ \frac{\widehat{MSE}^{(1)}(\mathbf{x}) - 0.07277}{0.00281 - 0.07277} & \text{if } 0.00281 < \widehat{MSE}^{(1)}(\mathbf{x}) < 0.07277 \\ 0 & \text{if } \widehat{MSE}^{(1)}(\mathbf{x}) \geq 0.07277 \end{cases}$$

and

$$d^{(2)} = h\left(\widehat{MSE}^{(2)}(\mathbf{x})\right) = \begin{cases} 1 & \text{if } \widehat{MSE}^{(2)}(\mathbf{x}) \leq 1.06 \\ \frac{\widehat{MSE}^{(2)}(\mathbf{x}) - 8.42}{1.06 - 8.42} & \text{if } 1.06 < \widehat{MSE}^{(2)}(\mathbf{x}) < 8.42 \\ 0 & \text{if } \widehat{MSE}^{(2)}(\mathbf{x}) \geq 8.42. \end{cases}$$

The overall desirability function is determined as follows:

$$D = \left(\prod_{i=1}^2 d^{(i)}\right)^{1/2} = \left[\prod_{i=1}^2 h\left(\widehat{MSE}^{(i)}(\mathbf{x})\right)\right]^{1/2}.$$

The optimum operating conditions are obtained by solving the following optimization prob-

2.8. Conclusions

lem:

$$\begin{cases} \text{Maximize} & D \\ \text{Subject to} & -1 \leq x_j \leq 1; j = 1, 2, 3. \end{cases}$$

Table 2.6 presents the results obtained.

Table 2.6: Optimal settings and related estimates.

Method	Statistic	Optimal results for $y^{(1)}$	Optimal results for $y^{(2)}$
DF based on MSE	\mathbf{x}_{opt}	(0.38916, -1, -0.98629)	(0.38916, -1, -0.98629)
	$\widehat{E}_{\mathbf{z}}(y^{(i)}(\mathbf{x}_{opt}, \mathbf{z}))$	1.39	1.41
	$\widehat{Var}_{\mathbf{z}}(y^{(i)}(\mathbf{x}_{opt}, \mathbf{z}))$	0.00358	0.0555
	$\widehat{MSE}^{(i)}(\mathbf{x}_{opt})$	0.158	2.05
	Desirability	0.82496	0.82496

Industrial problems commonly call for a multiresponse approach in robust design since practical applications routinely present more than one quality characteristic to be targeted. The desirability function based upon the mean square error is an excellent tool for analysing the data obtained from such an experiment.

2.8 Conclusions

We present a methodological approach for conducting robust designs and analysing the data obtained. An overview of different approaches used to conduct robust design and analyse the data is provided. We have presented the Taguchi approach commonly known as double orthogonal array design and the combined array design. The mean square error criterion and desirability function have been used as optimization procedures to finding the optimal conditions of the process.

To highlight the practical implementation of the double orthogonal array design, we have given an illustrative example on a chemical process where the objectives are to diminish the proportion of impurities and to reduce the variance of the process.

Focusing on the proportion of impurities, the mean square error approach gives the best results, and when the main interest is the reduction of variability in the process, the desirability function seems to be the best method of analysis.

When both, the reduction of proportion of impurities and reduction of variability are of utmost importance, the selection of the design and the method of analysis to be used must take into account the estimated value of the quality loss function. Based upon the quality loss function, we have given an illustrative example that highlights the practical importance

2.8. Conclusions

of conducting a designed experiment in order to reduce the overall cost of the production. In this example, the estimate of the expected quality loss of the designed experiment, \widehat{Q}_{doe} , ranges between 59.116 and 68.596 monetary units, while the overall difference that indicates the gain incurred by conducting the process under the optimal conditions, Δ , ranges between 607.77 and 616.53 monetary units.

The combined array design is illustrated by an example of a robust design conducted on the elastic element of a force transducer. This is an example of a robust design with several response variables. The objective of the design is to determine the optimal combinations of the design factors so that the hysteresis indicator is minimized and the non-linearity indicator is brought to 1. Data analysis is based on desirability function applied on mean square error functions.

Chapter 3

Signal-to-Noise Ratios: Statistical Tests for Pairwise Comparisons

3.1 Introduction

Robust parameter design is one of the most creative and effective tools in quality engineering. This tool works by identifying factor settings to reduce the variation in products or processes. Robust parameter design had been practised in Japan for many years before it was introduced to the US by its originator Genichi Taguchi in the mid-1980's (Bérubé and Wu (2000)). Robust parameter design is the operation of choosing settings for the design parameters of a product or manufacturing process to reduce sensitivity to noise. Noise is hard-to-control variability affecting performance (León et al. (1987)). One of the central ideas in Taguchi's approach to parameter design is the use of the performance criterion that he called Signal-to-noise ratio (SNR) for variation reduction and parameter optimization. The signal-to-noise is a performance measure that combines the mean response and variability (Myers et al. (1992)). The extend to which maximization of such criterion can be linked with minimization of quadratic loss was considered by León et al. (1987). The signal-to-noise ratio that is used depends on the goal of the experiment. The different goals are as follows:

1. The nominal-the-best: The experimenter wishes for the response to attain a certain target value.
2. The smaller-the-better: The experimenter is interested in minimizing the response.
3. The larger-the-better: The experimenter is interested in maximizing the response.

The signal-to-noise ratio has generated a lot of controversies as seen by the discussions on Box's paper (Box (1988)) and the panel discussions edited by Nair (1992).

The objective of this chapter is to propose a statistical test for pairwise comparisons of treatments in robust design based on signal-to-noise ratio criteria.

3.2. Brief Recall of the Taguchi Design

3.2 Brief Recall of the Taguchi Design

We recall the robust design in double orthogonal array structure as initiated by Taguchi. [Taguchi \(1991\)](#) recommends the following strategy for parameter design optimization. The set-up of the experiment is a cross array (also called inner-outer array or double orthogonal array design). The control factors are varied according to an orthogonal array called control array (or inner array) and for each setting of the control factors, the noise factors are varied systematically according to an orthogonal array called noise array (or outer array). The analysis of the inner array is made with the signal-to-noise ratio being the performance criterion. One seeks to maximize the signal-to-noise ratio.

Suppose that the quality characteristic, say y , of a product or a process, depends on p design factors x_1, \dots, x_p and q noise factors z_1, \dots, z_q . The responses y_{ij} are the combinations of the levels of the design factors ($i = 1, 2, \dots, n$) and the levels of the noise factors ($j = 1, 2, \dots, r$). The total number of runs required to conduct an experiment in this case is $n \times r$. The experimental structure of double orthogonal array design is represented by Figure 3.1.

Figure 3.1: Experimental structure of the inner-outer array design.

				z_1	z_{11}	z_{21}	\dots	z_{r1}		
				z_2	z_{12}	z_{22}	\dots	z_{r2}		
				\vdots	\vdots	\vdots	\vdots	\vdots		
				z_q	z_{1q}	z_{2q}	\dots	z_{rq}		
x_1	x_2	\dots	x_p	Observations				\bar{y}	s^2	SNR
x_{11}	x_{12}	\dots	x_{1p}	y_{11}	y_{12}	\dots	y_{1r}	\bar{y}_1	s_1^2	SNR ₁
x_{21}	x_{22}	\dots	x_{2p}	y_{21}	y_{22}	\dots	y_{2r}	\bar{y}_2	s_2^2	SNR ₂
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
x_{n1}	x_{n2}	\dots	x_{np}	y_{n1}	y_{n2}	\dots	y_{nr}	\bar{y}_n	s_n^2	SNR _n

3.3 Forms of Signal-to-noise Ratio (SNR)

Let y_1, y_2, \dots, y_n be a realization of iid random variables Y_1, Y_2, \dots, Y_n normally distributed with mean μ and variance σ^2 .

Considering the objective of the designed experiment, Taguchi defines three situations and assigns the corresponding expression of signal-to-noise ratio to each case ([Phadke \(1989\)](#)).

3.3.1 Signal-to-noise ratio for the nominal-the-best case

In many cases, it is of interest to achieve a certain target value for the response, say $y = T$, while the variation is minimum. Deviations in either direction are undesirable. In this case,

3.3. Forms of Signal-to-noise Ratio (SNR)

Taguchi recommends the following signal-to-noise ratio:

$$\text{SNR}_T = 10 \log_{10} \left(\frac{\mu^2}{\sigma^2} \right). \quad (3.1)$$

Its estimate is obtained as

$$\widehat{\text{SNR}}_T = 10 \log_{10} \left(\frac{\bar{y}^2}{s^2} \right), \quad (3.2)$$

where

- $\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$ is the average location for an experimental run, and
- $s^2 = \frac{1}{n-1} \sum_{i=1}^n (y_i - \bar{y})^2$ is the sample variance computed from the response values in the inner array.

Note that Equation (3.2) can be written as $\widehat{\text{SNR}}_T = 10 \log_{10}(\bar{y}^2) - 10 \log_{10}(s^2)$. Kacker (1985) pointed out that often in cases where the response variance and mean are independent, one or more factors (tuning or adjustment factors) can be used in order to eliminate response bias, that is, the adjustments result in $E(y) = T$. If one assumes an additive model, the loss function $E(y - T)^2$ reduces to $\text{Var}(y)$. As a result, the estimate of the signal-to-noise ratio reduces to $\widehat{\text{SNR}}_T = -10 \log(s^2)$. The factor levels in the inner array are chosen so that the estimate of the signal-to-noise ratio is maximized. Consequently, if the mean \bar{y} is set at a target value, then maximizing $\widehat{\text{SNR}}_T$ is equivalent to minimizing $\log_{10}(s^2)$ (Myers et al. (1992)). When the variation in the $\log_{10}(s^2)$ component is larger than the variation in the $\log(\bar{y}^2)$ component, $\widehat{\text{SNR}}_T$ is dominated by the variation in $\log_{10}(s^2)$. Therefore an analysis of the signal-to-noise ratio essentially reduces to an analysis of $\log_{10}(s^2)$ (Bérubé and Wu (2000)).

3.3.2 Signal-to-noise ratio for the smaller-the-better case

Taguchi treats this situation as if there is a target value of zero. As a result, the quadratic loss function $E(y - 0)^2$ leads to a performance criterion derived from $E(y)^2$. The expectation is taken over the distribution of the noise variables. One assumes that each design point in the control variables contains n runs in the noise variables from the outer array. As a result, the performance characteristic is based on $\frac{1}{n} \sum_{i=1}^n y_i^2$.

Taguchi makes use of the logarithmic transformation, and thus suggests as the appropriate signal-to-noise ratio which is used when the objective is to minimize the response while the variation is minimum:

$$\widehat{\text{SNR}}_S = -10 \log_{10} \left(\frac{1}{n} \sum_{i=1}^n y_i^2 \right), \quad (3.3)$$

where n denotes the number of replications per run.

3.4. Asymptotic Distributions of the Estimates of Signal-to-noise Ratios

3.3.3 Signal-to-noise ratio for the larger-the-better case

The following estimate of the signal-to-noise ratio is used when the interest is to maximize the response while the variation is minimum, Taguchi treats this case in the same fashion as the smaller the better case but uses the reciprocal, $1/y$. Hence, the estimate of the signal-to-noise ratio is given by

$$\widehat{\text{SNR}}_L = -10 \log_{10} \left(\frac{1}{n} \sum_{i=1}^n \frac{1}{y_i^2} \right), \quad (3.4)$$

where n denotes the number of replications per run.

3.4 Asymptotic Distributions of the Estimates of Signal-to-noise Ratios

In order to conduct the tests of hypothesis for pairwise comparisons of signal-to-noise ratios, it is important to know the distributions of the estimates of signal-to-noise ratios. The multivariate Delta theorem (DasGupta (2008)) is applied for determining the asymptotic distributions of the estimates of signal-to-noise ratios.

3.4.1 Case of the estimate of the signal-to-noise ratio for the nominal-the-best case

Result 3.1 Asymptotic distribution of $\widehat{\text{SNR}}_T$

Let y_1, y_2, \dots, y_n be realizations of iid random variables Y_1, Y_2, \dots, Y_n normally distributed with mean μ and variance σ^2 . Then the estimate of the signal-to-noise ratio for the nominal-the-best case, $\widehat{\text{SNR}}_T$, is asymptotically distributed as normal with mean $\mu_{\widehat{\text{SNR}}_T} = \left(\frac{10}{\ln 10} \right) \ln \left(\frac{\mu^2}{\sigma^2} \right)$ and variance $\sigma_{\widehat{\text{SNR}}_T}^2 = \left(\frac{10}{\ln 10} \right)^2 \left(\frac{4\sigma^2}{n^2\mu^2} + \frac{2}{n^2} \right)$.

Proof

The estimate of the signal-to-noise ratio for the nominal-the-best case, say $\widehat{\text{SNR}}_T$, can be written as

$$\widehat{\text{SNR}}_T = 10 \log_{10} \left(\frac{\bar{y}^2}{s^2} \right) = \left(\frac{10}{\ln 10} \right) \ln \left(\frac{\bar{y}^2}{s^2} \right). \quad (3.5)$$

Let $\boldsymbol{\theta} = (\mu, \sigma^2)$ be the vector of unknown parameters of the normal population such that the vector $\hat{\boldsymbol{\theta}} = (\bar{y}, s^2)$ is its estimator. We recall that the variance-covariance matrix of $\hat{\boldsymbol{\theta}}$ is given by

$$\text{Var}(\hat{\boldsymbol{\theta}}) = \begin{pmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\sigma^4}{n} \end{pmatrix}. \quad (3.6)$$

3.4. Asymptotic Distributions of the Estimates of Signal-to-noise Ratios

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a bivariate function such that

$$f(\boldsymbol{\theta}) = f(\mu, \sigma^2) = \ln \left(\frac{\mu^2}{\sigma^2} \right). \quad (3.7)$$

The corresponding partial derivatives respect to μ and σ^2 are, respectively,

$$\frac{\partial f(\boldsymbol{\theta})}{\partial \mu} = \frac{\partial}{\partial \mu} \ln \left(\frac{\mu^2}{\sigma^2} \right) = \frac{2}{\mu} \quad \text{and} \quad \frac{\partial f(\boldsymbol{\theta})}{\partial \sigma^2} = \frac{\partial}{\partial \sigma^2} \ln \left(\frac{\mu^2}{\sigma^2} \right) = -\frac{1}{\sigma^2}. \quad (3.8)$$

The gradient vector is

$$\nabla f(\boldsymbol{\theta}) = \begin{pmatrix} \frac{2}{\mu} \\ -\frac{1}{\sigma^2} \end{pmatrix}. \quad (3.9)$$

Applying the multivariate Delta theorem leads to

$$\sqrt{n} \left[\ln \left(\frac{\bar{y}^2}{s^2} \right) - \ln \left(\frac{\mu^2}{\sigma^2} \right) \right] \stackrel{a}{\sim} N \left(0, \begin{pmatrix} \frac{2}{\mu} \\ -\frac{1}{\sigma^2} \end{pmatrix}^T \begin{pmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\sigma^4}{n} \end{pmatrix} \begin{pmatrix} \frac{2}{\mu} \\ -\frac{1}{\sigma^2} \end{pmatrix} \right) \quad (3.10)$$

i.e.,

$$\sqrt{n} \left[\ln \left(\frac{\bar{y}^2}{s^2} \right) - \ln \left(\frac{\mu^2}{\sigma^2} \right) \right] \stackrel{a}{\sim} N \left(0, \frac{4\sigma^2}{n\mu^2} + \frac{2}{n} \right) \quad (3.11)$$

or equivalently

$$\ln \left(\frac{\bar{y}^2}{s^2} \right) \stackrel{a}{\sim} N \left(\ln \left(\frac{\mu^2}{\sigma^2} \right), \frac{4\sigma^2}{n^2\mu^2} + \frac{2}{n^2} \right). \quad (3.12)$$

It follows that

$$\widehat{\text{SNR}}_T \stackrel{a}{\sim} \left(\frac{10}{\ln 10} \right) N \left(\ln \left(\frac{\mu^2}{\sigma^2} \right), \frac{4\sigma^2}{n^2\mu^2} + \frac{2}{n^2} \right), \quad (3.13)$$

where $\stackrel{a}{\sim}$ stands for *asymptotically*.

Therefore, the estimate of the signal-to-noise ratio is asymptotical distributed as normal, this is,

$$\widehat{\text{SNR}}_T \stackrel{a}{\sim} N(\mu_{\widehat{\text{SNR}}_T}, \sigma_{\widehat{\text{SNR}}_T}^2), \quad (3.14)$$

where

$$\begin{aligned} \mu_{\widehat{\text{SNR}}_T} &= \left(\frac{10}{\ln 10} \right) \ln \left(\frac{\mu^2}{\sigma^2} \right) \quad \text{and} \\ \sigma_{\widehat{\text{SNR}}_T}^2 &= \left(\frac{10}{\ln 10} \right)^2 \left(\frac{4\sigma^2}{n^2\mu^2} + \frac{2}{n^2} \right) \quad \blacksquare \end{aligned} \quad (3.15)$$

3.4. Asymptotic Distributions of the Estimates of Signal-to-noise Ratios

3.4.2 Case of the estimate of the signal-to-noise ratio for the smaller-the-better case

Let $\widehat{\text{SNR}}_{S_1}$ and $\widehat{\text{SNR}}_{S_2}$ be the signal-to-noise ratios calculated from the samples y_1 and y_2 , respectively. The corresponding population signal-to-noise ratios are SNR_{S_1} and SNR_{S_2} , respectively. The hypotheses to be tested are

$$H_0 : \text{SNR}_{S_1} = \text{SNR}_{S_2} \text{ vs } H_1 : \text{SNR}_{S_1} \neq \text{SNR}_{S_2} \quad (3.16)$$

or equivalently

$$H_0 : \text{SNR}_{S_1} - \text{SNR}_{S_2} = 0 \text{ vs } H_1 : \text{SNR}_{S_1} - \text{SNR}_{S_2} \neq 0. \quad (3.17)$$

Result 3.2 Mean and standard deviation of $\widehat{\text{SNR}}_{S_1} - \widehat{\text{SNR}}_{S_2}$

Let $y_1 = y_{11}, y_{12}, \dots, y_{1n_1}$ and $y_2 = y_{21}, y_{22}, \dots, y_{2n_2}$ be two independent samples of sizes n_1 and n_2 , respectively, drawn from two independent normal populations with mean μ_i and variance σ_i^2 , $i = 1, 2$. Under H_0 , the mean and standard deviation of $\widehat{\text{SNR}}_{S_1} - \widehat{\text{SNR}}_{S_2}$ are asymptotically zero and $\left(\frac{-10}{\ln 10}\right) \sqrt{\frac{\sigma_1^2}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{8\mu_1^2\sigma_1^4}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{\sigma_2^2}{n_2^2(\sigma_2^2 + \mu_2^2)^2} + \frac{8\mu_2^2\sigma_2^4}{n_2^2(\sigma_2^2 + \mu_2^2)^2}}$, respectively.

Proof

In fact,

$$\begin{aligned} \mu_{\widehat{\text{SNR}}_{S_1} - \widehat{\text{SNR}}_{S_2}} &= \mu_{\widehat{\text{SNR}}_{S_1}} - \mu_{\widehat{\text{SNR}}_{S_2}} \\ &= \left(\frac{-10}{\ln 10}\right) \ln(\sigma_1^2 + \mu_1^2) - \left(\frac{-10}{\ln 10}\right) \ln(\sigma_2^2 + \mu_2^2) \\ &= \text{SNR}_{S_1} - \text{SNR}_{S_2} = 0. \end{aligned} \quad (3.18)$$

The standard deviation of the difference of $\widehat{\text{SNR}}_{S_1}$ and $\widehat{\text{SNR}}_{S_2}$, say $\sigma_{\widehat{\text{SNR}}_{S_1} - \widehat{\text{SNR}}_{S_2}}$, is determined as follows:

$$\begin{aligned} &\sigma_{\widehat{\text{SNR}}_{S_1} - \widehat{\text{SNR}}_{S_2}} \\ &= \sqrt{\sigma_{\widehat{\text{SNR}}_{S_1}}^2 + \sigma_{\widehat{\text{SNR}}_{S_2}}^2} \\ &= \sqrt{\left(\frac{-10}{\ln 10}\right)^2 \left[\left(\frac{\sigma_1^2}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{8\mu_1^2\sigma_1^4}{n_1^2(\sigma_1^2 + \mu_1^2)^2} \right) + \left(\frac{\sigma_2^2}{n_2^2(\sigma_2^2 + \mu_2^2)^2} + \frac{8\mu_2^2\sigma_2^4}{n_2^2(\sigma_2^2 + \mu_2^2)^2} \right) \right]} \\ &= \left(\frac{-10}{\ln 10}\right) \sqrt{\frac{\sigma_1^2}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{8\mu_1^2\sigma_1^4}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{\sigma_2^2}{n_2^2(\sigma_2^2 + \mu_2^2)^2} + \frac{8\mu_2^2\sigma_2^4}{n_2^2(\sigma_2^2 + \mu_2^2)^2}} \quad \blacksquare \end{aligned} \quad (3.19)$$

3.5. Proposed Approach for Comparing Groups Using the SNR

3.4.3 Case of the estimate of the signal-to-noise ratio for the larger-the-better case

Result 3.3 *Asymptotic distribution of \widehat{SNR}_L*

Let y_1, y_2, \dots, y_n be realizations of iid random variables Y_1, Y_2, \dots, Y_n normally distributed with mean μ and variance σ^2 . Then, the estimate of the signal-to-noise ratio for the smaller-the-better case, \widehat{SNR}_L , is asymptotically distributed as normal with mean

$$\mu_{\widehat{SNR}_L} = \left(\frac{-10}{\ln 10}\right) \ln(\sigma^2 + \mu^2) \text{ and variance } \sigma_{\widehat{SNR}_L}^2 = \left(\frac{-10}{\ln 10}\right)^2 \left(\frac{\sigma^2}{n^2(\sigma^2 + \mu^2)^2} + \frac{8\mu^2\sigma^4}{n^2(\sigma^2 + \mu^2)^2}\right).$$

Proof

We recall that the original samples are

$$\text{Sample 1: } y_1 = y_{11}, y_{12}, \dots, y_{1n_1} \text{ and Sample 2: } y_2 = y_{21}, y_{22}, \dots, y_{2n_2}.$$

Let $x_1 = \frac{1}{y_1}$ and $x_2 = \frac{1}{y_2}$ so that the new samples are

$$\text{New sample 1: } x_1 = x_{11}, x_{12}, \dots, x_{1n_1} \text{ and New sample 2: } x_2 = x_{21}, x_{22}, \dots, x_{2n_2}.$$

The estimate of the signal-to-noise ratio, \widehat{SNR}_L , can be written as

$$\begin{aligned} \widehat{SNR}_L &= -10 \log_{10} \left(\frac{1}{n} \sum_{i=1}^n x_i^2 \right) = -10 \log_{10} (E(x^2)) \\ &= -10 \log_{10} (s^2 + \bar{x}^2) \\ &= \left(\frac{-10}{\ln 10}\right) \ln (s^2 + \bar{x}^2). \end{aligned} \tag{3.20}$$

Following stepwise the same logic as in the case of signal-to-noise ratio for the smaller-the-better situation, we get

$$\widehat{SNR}_L \stackrel{a}{\sim} N(\mu_{\widehat{SNR}_L}, \sigma_{\widehat{SNR}_L}^2), \tag{3.21}$$

where

$$\begin{aligned} \mu_{\widehat{SNR}_L} &= \left(\frac{-10}{\ln 10}\right) \ln(\sigma^2 + \mu^2) \text{ and} \\ \sigma_{\widehat{SNR}_L}^2 &= \left(\frac{-10}{\ln 10}\right)^2 \left(\frac{\sigma^2}{n^2(\sigma^2 + \mu^2)^2} + \frac{8\mu^2\sigma^4}{n^2(\sigma^2 + \mu^2)^2}\right) \blacksquare \end{aligned} \tag{3.22}$$

3.5 Proposed Approach for Comparing Groups Using the SNR

Multiple comparisons of treatments is one of the most important topics in designed experiments. In the literature, the concept of multiple comparisons of treatments based on

3.5. Proposed Approach for Comparing Groups Using the SNR

signal-to-noise ratios is not studied. In this section, exploiting the properties of asymptotic normality and the Central Limit Theorem (Lehman and Romano (2005), Casella and Berger (2002)), we present a statistical test for pairwise comparisons of the values of signal-to-noise ratio. We begin by considering two independent normal populations with mean μ_i and variance σ_i^2 , $i = 1, 2$. Let SNR_1 and SNR_2 represent the corresponding signal-to-noise ratios. Suppose that y_1 and y_2 are two independent samples of sizes n_1 and n_2 , respectively, drawn from these two populations such that

$$\text{Sample 1: } y_1 = y_{11}, y_{12}, \dots, y_{1n_1} \text{ and Sample 2: } y_2 = y_{21}, y_{22}, \dots, y_{2n_2}.$$

The corresponding estimates of signal-to-noise ratios are $\widehat{\text{SNR}}_1$ and $\widehat{\text{SNR}}_2$, respectively. It is desired to test the hypothesis

$$H_0 : \text{SNR}_1 = \text{SNR}_2 \text{ against } H_1 : \text{SNR}_1 \neq \text{SNR}_2 \quad (3.23)$$

or equivalently,

$$H_0 : \text{SNR}_1 - \text{SNR}_2 = 0 \text{ against } H_1 : \text{SNR}_1 - \text{SNR}_2 \neq 0. \quad (3.24)$$

3.5.1 Proposed statistical test for pairwise comparisons of signal-to-noise ratios for the nominal-the-best case

In the situation where the response variable is of the nominal-the-best type, the hypotheses to be tested are

$$H_0 : \text{SNR}_{T_1} = \text{SNR}_{T_2} \text{ vs } H_1 : \text{SNR}_{T_1} \neq \text{SNR}_{T_2} \quad (3.25)$$

or equivalently,

$$H_0 : \text{SNR}_{T_1} - \text{SNR}_{T_2} = 0 \text{ vs } H_1 : \text{SNR}_{T_1} - \text{SNR}_{T_2} \neq 0. \quad (3.26)$$

Result 3.4 Mean and standard deviation of $\widehat{\text{SNR}}_{T_1} - \widehat{\text{SNR}}_{T_2}$

Let $y_1 = y_{11}, y_{12}, \dots, y_{1n_1}$ and $y_2 = y_{21}, y_{22}, \dots, y_{2n_2}$ be two independent samples of sizes n_1 and n_2 , respectively, drawn from two independent normal populations with mean μ_i and variance σ_i^2 , $i = 1, 2$. Under H_0 , the mean and standard deviation of $\widehat{\text{SNR}}_{T_1} - \widehat{\text{SNR}}_{T_2}$ are asymptotically zero and $\left(\frac{10}{\ln 10}\right) \sqrt{\frac{4\sigma_1^2}{n_1^2\mu_1^2} + \frac{2}{n_1^2} + \frac{4\sigma_2^2}{n_2^2\mu_2^2} + \frac{2}{n_2^2}}$, respectively.

Proof

In fact,

$$\begin{aligned} \mu_{\widehat{\text{SNR}}_{T_1} - \widehat{\text{SNR}}_{T_2}} &= \mu_{\widehat{\text{SNR}}_{T_1}} - \mu_{\widehat{\text{SNR}}_{T_2}} \\ &= \left(\frac{10}{\ln 10}\right) \ln \left(\frac{\mu_1^2}{\sigma_1^2}\right) - \left(\frac{10}{\ln 10}\right) \ln \left(\frac{\mu_2^2}{\sigma_2^2}\right) = \text{SNR}_{T_1} - \text{SNR}_{T_2} = 0. \end{aligned} \quad (3.27)$$

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The standard deviation of the difference of $\widehat{\text{SNR}}_{T_1}$ and $\widehat{\text{SNR}}_{T_2}$, say $\sigma_{\widehat{\text{SNR}}_{T_1} - \widehat{\text{SNR}}_{T_2}}$, is determined as follows:

$$\sigma_{\widehat{\text{SNR}}_{T_1} - \widehat{\text{SNR}}_{T_2}} = \sqrt{\sigma_{\widehat{\text{SNR}}_{T_1}}^2 + \sigma_{\widehat{\text{SNR}}_{T_2}}^2} = \sqrt{\left(\frac{10}{\ln 10}\right)^2 \left(\frac{4\sigma_1^2}{n_1^2 \mu_1^2} + \frac{2}{n_1^2}\right) + \left(\frac{10}{\ln 10}\right)^2 \left(\frac{4\sigma_2^2}{n_2^2 \mu_2^2} + \frac{2}{n_2^2}\right)} \quad (3.28)$$

$$= \left(\frac{10}{\ln 10}\right) \sqrt{\frac{4\sigma_1^2}{n_1^2 \mu_1^2} + \frac{2}{n_1^2} + \frac{4\sigma_2^2}{n_2^2 \mu_2^2} + \frac{2}{n_2^2}} \quad \blacksquare \quad (3.29)$$

Result 3.5 Statistical test for comparing SNR_{T_1} and SNR_{T_2}

The statistical test for comparing SNR_{T_1} and SNR_{T_2} in the case μ_1, μ_2, σ_1 and σ_2 are known is

$$z = \frac{\ln\left(\frac{\bar{y}_1^2}{s_1^2}\right) - \ln\left(\frac{\bar{y}_2^2}{s_2^2}\right)}{\sqrt{\frac{4\sigma_1^2}{n_1^2 \mu_1^2} + \frac{2}{n_1^2} + \frac{4\sigma_2^2}{n_2^2 \mu_2^2} + \frac{2}{n_2^2}}}, \text{ and the statistical test becomes } t = \frac{\ln\left(\frac{\bar{y}_1^2}{s_1^2}\right) - \ln\left(\frac{\bar{y}_2^2}{s_2^2}\right)}{\sqrt{\frac{4s_1^2}{n_1^2 \bar{y}_1^2} + \frac{2}{n_1^2} + \frac{4s_2^2}{n_2^2 \bar{y}_2^2} + \frac{2}{n_2^2}}} \text{ when } \mu_1, \mu_2, \sigma_1$$

and σ_2 are unknown.

Proof

The statistical test in case μ_1, μ_2, σ_1 and σ_2 are known is given by

$$z = \frac{\left(\widehat{\text{SNR}}_{T_1} - \widehat{\text{SNR}}_{T_2}\right) - (\text{SNR}_{T_1} - \text{SNR}_{T_2})}{\sigma_{\widehat{\text{SNR}}_{T_1} - \widehat{\text{SNR}}_{T_2}}}, \quad (3.30)$$

and the statistical test when μ_1, μ_2, σ_1 and σ_2 are unknown is

$$t = \frac{\left(\widehat{\text{SNR}}_{T_1} - \widehat{\text{SNR}}_{T_2}\right) - (\text{SNR}_{T_1} - \text{SNR}_{T_2})}{\widehat{\sigma}_{\widehat{\text{SNR}}_{T_1} - \widehat{\text{SNR}}_{T_2}}}. \quad (3.31)$$

Under H_0 , $\text{SNR}_{T_1} - \text{SNR}_{T_2} = 0$, and these statistics reduce to the following expressions.

The statistical test in case μ_1, μ_2, σ_1 and σ_2 are known is given by

$$\begin{aligned} z &= \frac{\widehat{\text{SNR}}_{T_1} - \widehat{\text{SNR}}_{T_2}}{\sigma_{\widehat{\text{SNR}}_{T_1} - \widehat{\text{SNR}}_{T_2}}} = \frac{\left(\frac{10}{\ln 10}\right) \ln\left(\frac{\bar{y}_1^2}{s_1^2}\right) - \left(\frac{10}{\ln 10}\right) \ln\left(\frac{\bar{y}_2^2}{s_2^2}\right)}{\left(\frac{10}{\ln 10}\right) \sqrt{\frac{4\sigma_1^2}{n_1^2 \mu_1^2} + \frac{2}{n_1^2} + \frac{4\sigma_2^2}{n_2^2 \mu_2^2} + \frac{2}{n_2^2}}} \\ &= \frac{\ln\left(\frac{\bar{y}_1^2}{s_1^2}\right) - \ln\left(\frac{\bar{y}_2^2}{s_2^2}\right)}{\sqrt{\frac{4\sigma_1^2}{n_1^2 \mu_1^2} + \frac{2}{n_1^2} + \frac{4\sigma_2^2}{n_2^2 \mu_2^2} + \frac{2}{n_2^2}}}. \end{aligned} \quad (3.32)$$

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The statistical test when μ_1, μ_2, σ_1 and σ_2 are unknown is

$$\begin{aligned}
 t &= \frac{\widehat{\text{SNR}}_{T_1} - \widehat{\text{SNR}}_{T_2}}{\widehat{\sigma}_{\widehat{\text{SNR}}_{T_1} - \widehat{\text{SNR}}_{T_2}}} = \frac{\left(\frac{10}{\ln 10}\right) \ln\left(\frac{\bar{y}_1^2}{s_1^2}\right) - \left(\frac{10}{\ln 10}\right) \ln\left(\frac{\bar{y}_2^2}{s_2^2}\right)}{\left(\frac{10}{\ln 10}\right) \sqrt{\frac{4s_1^2}{n_1^2 \bar{y}_1^2} + \frac{2}{n_1^2} + \frac{4s_2^2}{n_2^2 \bar{y}_2^2} + \frac{2}{n_2^2}}} \\
 &= \frac{\ln\left(\frac{\bar{y}_1^2}{s_1^2}\right) - \ln\left(\frac{\bar{y}_2^2}{s_2^2}\right)}{\sqrt{\frac{4s_1^2}{n_1^2 \bar{y}_1^2} + \frac{2}{n_1^2} + \frac{4s_2^2}{n_2^2 \bar{y}_2^2} + \frac{2}{n_2^2}}} \quad \blacksquare \quad (3.33)
 \end{aligned}$$

Under H_0 , $z \stackrel{a}{\sim} N(0, 1)$ and $t \stackrel{a}{\sim} t_{n_1+n_2-2}$, where $\nu = n_1 + n_2 - 2$ represents the degrees of freedom of the t distribution. The null hypothesis, H_0 , is rejected if $|z| > z_{\alpha/2}$ or $|t| > t_{\alpha/2, \nu}$, where $z_{\alpha/2}$ is the $\alpha/2$ quantile of the standard normal distribution and $t_{\alpha/2, \nu}$ is the $\alpha/2$ quantile of the t distribution with ν degrees of freedom.

3.5.2 Proposed statistical test for pairwise comparisons of signal-to-noise ratios for the smaller-the-better case

Result 3.6 Statistical test for comparing SNR_{S_1} and SNR_{S_2}

The statistical test for comparing SNR_{S_1} and SNR_{S_2} in the case μ_1, μ_2, σ_1 and σ_2 are known

is $z = \frac{\ln\left(\frac{s_2^2 + \bar{y}_2^2}{s_1^2 + \bar{y}_1^2}\right)}{\sqrt{\frac{\sigma_1^2}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{8\mu_1^2\sigma_1^4}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{\sigma_2^2}{n_2^2(\sigma_2^2 + \mu_2^2)^2} + \frac{8\mu_2^2\sigma_2^4}{n_2^2(\sigma_2^2 + \mu_2^2)^2}}}$, and the statistical test becomes

$$t = \frac{\ln\left(\frac{s_2^2 + \bar{y}_2^2}{s_1^2 + \bar{y}_1^2}\right)}{\sqrt{\frac{s_1^2}{n_1^2(s_1^2 + \bar{y}_1^2)^2} + \frac{8\bar{y}_1^2 s_1^4}{n_1^2(s_1^2 + \bar{y}_1^2)^2} + \frac{s_2^2}{n_2^2(s_2^2 + \bar{y}_2^2)^2} + \frac{8\bar{y}_2^2 s_2^4}{n_2^2(s_2^2 + \bar{y}_2^2)^2}}}, \text{ when } \mu_1, \mu_2, \sigma_1 \text{ and } \sigma_2 \text{ are unknown.}$$

Proof

The statistical test in case μ_1, μ_2, σ_1 and σ_2 are known is given by

$$z = \frac{\left(\widehat{\text{SNR}}_{S_1} - \widehat{\text{SNR}}_{S_2}\right) - (\text{SNR}_{S_1} - \text{SNR}_{S_2})}{\sigma_{\widehat{\text{SNR}}_{S_1} - \widehat{\text{SNR}}_{S_2}}}, \quad (3.34)$$

and the statistical test when μ_1, μ_2, σ_1 and σ_2 are unknown is

$$t = \frac{\left(\widehat{\text{SNR}}_{S_1} - \widehat{\text{SNR}}_{S_2}\right) - (\text{SNR}_{S_1} - \text{SNR}_{S_2})}{\widehat{\sigma}_{\widehat{\text{SNR}}_{S_1} - \widehat{\text{SNR}}_{S_2}}}. \quad (3.35)$$

3.5. Proposed Approach for Comparing Groups Using the SNR

Under H_0 , $\text{SNR}_{S_1} - \text{SNR}_{S_2} = 0$, and the statistical tests are reduced into the following forms. The statistical test in case μ_1, μ_2, σ_1 and σ_2 are known is given by

$$\begin{aligned}
 z &= \frac{\widehat{\text{SNR}}_{S_1} - \widehat{\text{SNR}}_{S_2}}{\widehat{\sigma}_{\widehat{\text{SNR}}_{S_1} - \widehat{\text{SNR}}_{S_2}}} = \frac{\left(\frac{10}{\ln 10}\right) \ln(s_1^2 + \bar{y}_1^2) - \left(\frac{10}{\ln 10}\right) \ln(s_2^2 + \bar{y}_2^2)}{\left(\frac{-10}{\ln 10}\right) \sqrt{\frac{\sigma_1^2}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{8\mu_1^2\sigma_1^4}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{\sigma_2^2}{n_2^2(\sigma_2^2 + \mu_2^2)^2} + \frac{8\mu_2^2\sigma_2^4}{n_2^2(\sigma_2^2 + \mu_2^2)^2}}} \\
 &= \frac{\ln(s_2^2 + \bar{y}_2^2) - \ln(s_1^2 + \bar{y}_1^2)}{\sqrt{\frac{\sigma_1^2}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{8\mu_1^2\sigma_1^4}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{\sigma_2^2}{n_2^2(\sigma_2^2 + \mu_2^2)^2} + \frac{8\mu_2^2\sigma_2^4}{n_2^2(\sigma_2^2 + \mu_2^2)^2}}} \\
 &= \frac{\ln\left(\frac{s_2^2 + \bar{y}_2^2}{s_1^2 + \bar{y}_1^2}\right)}{\sqrt{\frac{\sigma_1^2}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{8\mu_1^2\sigma_1^4}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{\sigma_2^2}{n_2^2(\sigma_2^2 + \mu_2^2)^2} + \frac{8\mu_2^2\sigma_2^4}{n_2^2(\sigma_2^2 + \mu_2^2)^2}}}. \tag{3.36}
 \end{aligned}$$

The statistical test when μ_1, μ_2, σ_1 and σ_2 are unknown is

$$\begin{aligned}
 t &= \frac{\widehat{\text{SNR}}_{S_1} - \widehat{\text{SNR}}_{S_2}}{\widehat{\sigma}_{\widehat{\text{SNR}}_{S_1} - \widehat{\text{SNR}}_{S_2}}} = \frac{\left(\frac{10}{\ln 10}\right) \ln(s_1^2 + \bar{y}_1^2) - \left(\frac{10}{\ln 10}\right) \ln(s_2^2 + \bar{y}_2^2)}{\left(\frac{-10}{\ln 10}\right) \sqrt{\frac{s_1^2}{n_1^2(s_1^2 + \bar{y}_1^2)^2} + \frac{8\bar{y}_1^2 s_1^4}{n_1^2(s_1^2 + \bar{y}_1^2)^2} + \frac{s_2^2}{n_2^2(s_2^2 + \bar{y}_2^2)^2} + \frac{8\bar{y}_2^2 s_2^4}{n_2^2(s_2^2 + \bar{y}_2^2)^2}}} \\
 &= \frac{\ln(s_2^2 + \bar{y}_2^2) - \ln(s_1^2 + \bar{y}_1^2)}{\sqrt{\frac{s_1^2}{n_1^2(s_1^2 + \bar{y}_1^2)^2} + \frac{8\bar{y}_1^2 s_1^4}{n_1^2(s_1^2 + \bar{y}_1^2)^2} + \frac{s_2^2}{n_2^2(s_2^2 + \bar{y}_2^2)^2} + \frac{8\bar{y}_2^2 s_2^4}{n_2^2(s_2^2 + \bar{y}_2^2)^2}}} \\
 &= \frac{\ln\left(\frac{s_2^2 + \bar{y}_2^2}{s_1^2 + \bar{y}_1^2}\right)}{\sqrt{\frac{s_1^2}{n_1^2(s_1^2 + \bar{y}_1^2)^2} + \frac{8\bar{y}_1^2 s_1^4}{n_1^2(s_1^2 + \bar{y}_1^2)^2} + \frac{s_2^2}{n_2^2(s_2^2 + \bar{y}_2^2)^2} + \frac{8\bar{y}_2^2 s_2^4}{n_2^2(s_2^2 + \bar{y}_2^2)^2}}} \blacksquare \tag{3.37}
 \end{aligned}$$

As mentioned before, under H_0 , $z \stackrel{a}{\sim} N(0, 1)$ and $t \stackrel{a}{\sim} t_{n_1+n_2-2}$, where $\nu = n_1 + n_2 - 2$ represents the degrees of freedom of the t distribution. The null hypothesis, H_0 , is rejected if $|z| > z_{\alpha/2}$ or $|t| > t_{\alpha/2, \nu}$, where $z_{\alpha/2}$ is the $\alpha/2$ quantile of the standard normal distribution and $t_{\alpha/2, \nu}$ is the $\alpha/2$ quantile of the t distribution with ν degrees of freedom.

3.5.3 Proposed statistical test for pairwise comparisons of signal-to-noise ratios for the larger-the-better case

Let $\widehat{\text{SNR}}_{L_1}$ and $\widehat{\text{SNR}}_{L_2}$ be the estimates of the signal-to-noise ratios calculated from the transformed samples x_1 and x_2 respectively. The corresponding population signal-to-noise ratios are SNR_{L_1} and SNR_{L_2} , respectively. The hypotheses to be tested are

$$H_0 : \text{SNR}_{L_1} = \text{SNR}_{L_2} \text{ vs } H_1 : \text{SNR}_{L_1} \neq \text{SNR}_{L_2} \tag{3.38}$$

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or equivalently

$$H_0 : \text{SNR}_{L_1} - \text{SNR}_{L_2} = 0 \text{ vs } H_1 : \text{SNR}_{L_1} - \text{SNR}_{L_2} \neq 0. \quad (3.39)$$

Result 3.7 Mean and standard deviation of $\widehat{\text{SNR}}_{L_1} - \widehat{\text{SNR}}_{L_2}$

Let $y_1 = y_{11}, y_{12}, \dots, y_{1n_1}$ and $y_2 = y_{21}, y_{22}, \dots, y_{2n_2}$ be two independent samples of sizes n_1 and n_2 , respectively, drawn from two independent normal populations with mean μ_i and variance $\sigma_i^2, i = 1, 2$. Under H_0 , the mean and standard deviation of $\widehat{\text{SNR}}_{L_1} - \widehat{\text{SNR}}_{L_2}$ are asymptotically zero and $\left(\frac{-10}{\ln 10}\right) \sqrt{\frac{\sigma_1^2}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{8\mu_1^2\sigma_1^4}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{\sigma_2^2}{n_2^2(\sigma_2^2 + \mu_2^2)^2} + \frac{8\mu_2^2\sigma_2^4}{n_2^2(\sigma_2^2 + \mu_2^2)^2}}$, respectively.

Proof

By analogy to the case of the smaller-the-better situation, under H_0 , the mean of the difference of $\widehat{\text{SNR}}_{L_1}$ and $\widehat{\text{SNR}}_{L_2}$ is asymptotically zero.

The standard deviation of the difference of $\widehat{\text{SNR}}_{L_1}$ and $\widehat{\text{SNR}}_{L_2}$, say $\sigma_{\widehat{\text{SNR}}_{L_1} - \widehat{\text{SNR}}_{L_2}}$, is calculated as follows:

$$\begin{aligned} & \sigma_{\widehat{\text{SNR}}_{L_1} - \widehat{\text{SNR}}_{L_2}} \\ &= \sqrt{\sigma_{\widehat{\text{SNR}}_{S_1}}^2 + \sigma_{\widehat{\text{SNR}}_{S_2}}^2} \\ &= \left(\frac{-10}{\ln 10}\right) \sqrt{\frac{\sigma_1^2}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{8\mu_1^2\sigma_1^4}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{\sigma_2^2}{n_2^2(\sigma_2^2 + \mu_2^2)^2} + \frac{8\mu_2^2\sigma_2^4}{n_2^2(\sigma_2^2 + \mu_2^2)^2}} \blacksquare \end{aligned} \quad (3.40)$$

Result 3.8 Statistical test for comparing SNR_{L_1} and SNR_{L_2}

The statistical test for comparing SNR_{L_1} and SNR_{L_2} in the case μ_1, μ_2, σ_1 and σ_2 are known

$$\text{is } z = \frac{\ln\left(\frac{s_2^2 + \bar{x}_2^2}{s_1^2 + \bar{x}_1^2}\right)}{\sqrt{\frac{\sigma_1^2}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{8\mu_1^2\sigma_1^4}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{\sigma_2^2}{n_2^2(\sigma_2^2 + \mu_2^2)^2} + \frac{8\mu_2^2\sigma_2^4}{n_2^2(\sigma_2^2 + \mu_2^2)^2}}}, \text{ and the statistical test becomes}$$

$$t = \frac{\ln\left(\frac{s_2^2 + \bar{x}_2^2}{s_1^2 + \bar{x}_1^2}\right)}{\sqrt{\frac{s_1^2}{n_1^2(s_1^2 + \bar{x}_1^2)^2} + \frac{8\bar{x}_1^2 s_1^4}{n_1^2(s_1^2 + \bar{x}_1^2)^2} + \frac{s_2^2}{n_2^2(s_2^2 + \bar{x}_2^2)^2} + \frac{8\bar{x}_2^2 s_2^4}{n_2^2(s_2^2 + \bar{x}_2^2)^2}}}, \text{ when } \mu_1, \mu_2, \sigma_1 \text{ and } \sigma_2 \text{ are unknown.}$$

Proof

The statistical test in case μ_1, μ_2, σ_1 and σ_2 are known is given by

$$z = \frac{\left(\widehat{\text{SNR}}_{L_1} - \widehat{\text{SNR}}_{L_2}\right) - (\text{SNR}_{L_1} - \text{SNR}_{L_2})}{\sigma_{\widehat{\text{SNR}}_{L_1} - \widehat{\text{SNR}}_{L_2}}}, \quad (3.41)$$

3.6. Monte Carlo Study of Properties of Proposed Tests

and the statistical test when μ_1, μ_2, σ_1 and σ_2 are unknown is

$$t = \frac{(\widehat{\text{SNR}}_{L_1} - \widehat{\text{SNR}}_{L_2}) - (\text{SNR}_{L_1} - \text{SNR}_{L_2})}{\widehat{\sigma}_{\widehat{\text{SNR}}_{L_1} - \widehat{\text{SNR}}_{L_2}}}. \quad (3.42)$$

Under H_0 , $\text{SNR}_{L_1} - \text{SNR}_{L_2} = 0$. Therefore, the statistical test in case μ_1, μ_2, σ_1 and σ_2 are known is given by

$$\begin{aligned} z &= \frac{\widehat{\text{SNR}}_{L_1} - \widehat{\text{SNR}}_{L_2}}{\sigma_{\widehat{\text{SNR}}_{L_1} - \widehat{\text{SNR}}_{L_2}}} = \frac{\left(\frac{10}{\ln 10}\right) \ln(s_1^2 + \bar{x}_1^2) - \left(\frac{10}{\ln 10}\right) \ln(s_2^2 + \bar{x}_2^2)}{\left(\frac{-10}{\ln 10}\right) \sqrt{\frac{\sigma_1^2}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{8\mu_1^2\sigma_1^4}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{\sigma_2^2}{n_2^2(\sigma_2^2 + \mu_2^2)^2} + \frac{8\mu_2^2\sigma_2^4}{n_2^2(\sigma_2^2 + \mu_2^2)^2}}} \\ &= \frac{\ln(s_2^2 + \bar{x}_2^2) - \ln(s_1^2 + \bar{x}_1^2)}{\sqrt{\frac{\sigma_1^2}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{8\mu_1^2\sigma_1^4}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{\sigma_2^2}{n_2^2(\sigma_2^2 + \mu_2^2)^2} + \frac{8\mu_2^2\sigma_2^4}{n_2^2(\sigma_2^2 + \mu_2^2)^2}}} \\ &= \frac{\ln\left(\frac{s_2^2 + \bar{x}_2^2}{s_1^2 + \bar{x}_1^2}\right)}{\sqrt{\frac{\sigma_1^2}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{8\mu_1^2\sigma_1^4}{n_1^2(\sigma_1^2 + \mu_1^2)^2} + \frac{\sigma_2^2}{n_2^2(\sigma_2^2 + \mu_2^2)^2} + \frac{8\mu_2^2\sigma_2^4}{n_2^2(\sigma_2^2 + \mu_2^2)^2}}}. \end{aligned} \quad (3.43)$$

The statistical test when μ_1, μ_2, σ_1 and σ_2 are unknown is

$$\begin{aligned} t &= \frac{\widehat{\text{SNR}}_{L_1} - \widehat{\text{SNR}}_{L_2}}{\widehat{\sigma}_{\widehat{\text{SNR}}_{L_1} - \widehat{\text{SNR}}_{L_2}}} = \frac{\left(\frac{10}{\ln 10}\right) \ln(s_1^2 + \bar{x}_1^2) - \left(\frac{10}{\ln 10}\right) \ln(s_2^2 + \bar{x}_2^2)}{\left(\frac{-10}{\ln 10}\right) \sqrt{\frac{s_1^2}{n_1^2(s_1^2 + \bar{x}_1^2)^2} + \frac{8\bar{x}_1^2 s_1^4}{n_1^2(s_1^2 + \bar{x}_1^2)^2} + \frac{s_2^2}{n_2^2(s_2^2 + \bar{x}_2^2)^2} + \frac{8\bar{x}_2^2 s_2^4}{n_2^2(s_2^2 + \bar{x}_2^2)^2}}} \\ &= \frac{\ln(s_2^2 + \bar{x}_2^2) - \ln(s_1^2 + \bar{x}_1^2)}{\sqrt{\frac{s_1^2}{n_1^2(s_1^2 + \bar{x}_1^2)^2} + \frac{8\bar{x}_1^2 s_1^4}{n_1^2(s_1^2 + \bar{x}_1^2)^2} + \frac{s_2^2}{n_2^2(s_2^2 + \bar{x}_2^2)^2} + \frac{8\bar{x}_2^2 s_2^4}{n_2^2(s_2^2 + \bar{x}_2^2)^2}}} \\ &= \frac{\ln\left(\frac{s_2^2 + \bar{x}_2^2}{s_1^2 + \bar{x}_1^2}\right)}{\sqrt{\frac{s_1^2}{n_1^2(s_1^2 + \bar{x}_1^2)^2} + \frac{8\bar{x}_1^2 s_1^4}{n_1^2(s_1^2 + \bar{x}_1^2)^2} + \frac{s_2^2}{n_2^2(s_2^2 + \bar{x}_2^2)^2} + \frac{8\bar{x}_2^2 s_2^4}{n_2^2(s_2^2 + \bar{x}_2^2)^2}}}. \quad \blacksquare \end{aligned} \quad (3.44)$$

Under H_0 , $z \stackrel{a}{\sim} N(0, 1)$ and $t \stackrel{a}{\sim} t_{n_1 + n_2 - 2}$, where $\nu = n_1 + n_2 - 2$ represents the degrees of freedom of the t distribution. The null hypothesis, H_0 , is rejected if $|z| > z_{\alpha/2}$ or $|t| > t_{\alpha/2, \nu}$, where $z_{\alpha/2}$ is the $\alpha/2$ quantile of the standard normal distribution and $t_{\alpha/2, \nu}$ is the $\alpha/2$ quantile of the t distribution with ν degrees of freedom.

3.6 Monte Carlo Study of Properties of Proposed Tests

Monte Carlo simulations are performed to evaluate the performance of the proposed statistical tests in terms of test sizes and powers. Sample means and sample variances are used to determine the estimates of signal-to-noise ratios. Simulation under H_0 , this is, simulation

3.6. Monte Carlo Study of Properties of Proposed Tests

with equal population parameters ($\mu_X = \mu_Y$ and $\sigma_X = \sigma_Y$) permit estimating the test size. Under H_1 , simulations are conducted after applying an increment Δ to the population parameters. Simulations with different values of population parameters give the estimates of power tests.

3.6.1 Procedure for Monte Carlo simulation

Simulation process has been conducted according to the following procedure:

1. From two independent normal populations, X and Y , such that $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$, simulate two independent samples of sizes $n_X = n_Y = 10$.
2. Calculate sample means and sample variances; \bar{X}, \bar{Y}, s_X^2 and s_Y^2 .
3. Calculate the estimates of signal-to-noise ratios; \widehat{SNR}_X and \widehat{SNR}_Y .
4. Based on asymptotic normality of the estimates of signal-to-noise ratio, simulate $MC = 10000$ replicates of

$$\widehat{SNR}_X \overset{a}{\sim} N\left(\mu_{\widehat{SNR}_X}, \sigma_{\widehat{SNR}_X}^2\right)$$

and

$$\widehat{SNR}_Y \overset{a}{\sim} N\left(\mu_{\widehat{SNR}_Y}, \sigma_{\widehat{SNR}_Y}^2\right).$$

Four different configurations of sample sizes are used: $n = 10, 20, 30, 60$. These sample sizes are generated using the function $u + e^i$, $u = 4, i = 1, 2, 3, 4$ rounded to the nearest multiple of 10.

5. For each replicate, conduct a t test for the null hypothesis $H_0 : SNR_X - SNR_Y = 0$, and count the number of rejections (#Rejections).
6. Determine the rejection rate: $\frac{\#Rejections}{MC}$.

The parameters used in Step 1 are determined by applying an increment Δ according to the following scheme:

1. Simultaneous change of population means and population variances. Population parameters are determined as follows:
 $\mu_Y = \mu_X + \Delta_\mu, \sigma_Y = \sigma_X + \Delta_\sigma,$
 Δ_μ and Δ_σ are increments in population mean and population variance, respectively.
2. Changing population means and maintaining population variances at constant values. Population parameters are determined as follows:
 $\mu_Y = \mu_X + \Delta_\mu, \sigma_X = \sigma_Y.$

3.6. Monte Carlo Study of Properties of Proposed Tests

Table 3.1: Estimated Type I error rates of t test for various sample sizes.

Sample size	SNR_T	SNR_S	SNR_L
10	0.0499	0.0538	0.0499
20	0.0500	0.0462	0.0500
30	0.0498	0.0494	0.0498
60	0.0496	0.0510	0.0496

3. Changing population variances and maintaining population means at constant values. Population parameters are determined as follows:

$$\sigma_Y = \sigma_X + \Delta_\sigma, \mu_X = \mu_Y.$$

Four different configurations of increments are used: $\Delta = 0.001, 0.01, 0.1, 1$. Increment $\Delta = 0$ implies equal parameters.

Table 3.1 shows the estimated sizes of the test statistics for all three cases of signal-to-noise ratio. The population parameters used are $\mu_X = \mu_Y = 35$, $\sigma_X = \sigma_Y = 2$. The row entries represent the proportion of times H_0 was rejected at $\alpha = 0.05$ under H_0 , this is, the proportion of times H_0 is wrongly rejected. Regardless of the types of response variable, these are, the nominal-the-best, the smaller-the-better and the larger-the-better, the test size is very close to the significance level. Moreover, it seems that the sample size does not affect the value of the test size.

Table 3.2 contains the estimated powers obtained in changing the population means and population variances simultaneously. In this case, the population parameters used in simulations are:

$$\mu_Y = \mu_X + \Delta_\mu, \sigma_Y = \sigma_X + \Delta_\sigma.$$

The row entries represent the proportion of times H_0 is rejected at $\alpha = 0.05$ under H_1 , this is, the proportion of times H_0 is correctly rejected.

Table 3.3 contains the estimated powers, obtained in changing the population means and maintaining population variances at constant values. In this case,

$$\mu_Y = \mu_X + \Delta_\mu, \sigma_X = \sigma_Y.$$

The row entries represent the proportion of times H_0 is rejected at $\alpha = 0.05$ under H_1 .

3.6. Monte Carlo Study of Properties of Proposed Tests

Table 3.2: Estimated powers of t test for various sample sizes and various increments, changing the population means and population variances simultaneously.

(a) Signal-to-noise ratio for the nominal-the-best case

Sample size	$\Delta_\mu = 0.001$	$\Delta_\mu = 0.01$	$\Delta_\mu = 0.1$	$\Delta_\mu = 1$
	$\Delta_\sigma = 0.001$	$\Delta_\sigma = 0.01$	$\Delta_\sigma = 0.1$	$\Delta_\sigma = 1$
10	0.0503	0.0799	0.9990	1
20	0.0589	0.8642	1	1
30	0.1365	1	1	1
60	0.9981	1	1	1

(b) Signal-to-noise ratio for the smaller-the-better case

Sample size	$\Delta_\mu = 0.001$	$\Delta_\mu = 0.01$	$\Delta_\mu = 0.1$	$\Delta_\mu = 1$
	$\Delta_\sigma = 0.001$	$\Delta_\sigma = 0.01$	$\Delta_\sigma = 0.1$	$\Delta_\sigma = 1$
10	0.0532	0.0622	0.7140	1
20	0.0486	0.3666	1	1
30	0.0760	0.9950	1	1
60	0.7392	1	1	1

(c) Signal-to-noise ratio for the larger-the-better case

Sample size	$\Delta_\mu = 0.001$	$\Delta_\mu = 0.01$	$\Delta_\mu = 0.1$	$\Delta_\mu = 1$
	$\Delta_\sigma = 0.001$	$\Delta_\sigma = 0.01$	$\Delta_\sigma = 0.1$	$\Delta_\sigma = 1$
10	0.0498	0.0589	0.7177	1
20	0.0524	0.3690	1	1
30	0.0755	0.9950	1	1
60	0.7412	1	1	1

3.6. Monte Carlo Study of Properties of Proposed Tests

Table 3.3: Estimated powers of t test for various sample sizes and various increments, obtained in changing the population means and maintaining population variance at constant values.

(a) Signal-to-noise ratio for the nominal-the-best case

Sample size	$\Delta_\mu = 0.001$	$\Delta_\mu = 0.01$	$\Delta_\mu = 0.1$	$\Delta_\mu = 1$
10	0.0499	0.0504	0.0622	1
20	0.0497	0.0549	0.4549	1
30	0.0503	0.0800	0.9996	1
60	0.0596	0.08449	1	1

(b) Signal-to-noise ratio for the smaller-the-better case

Sample size	$\Delta_\mu = 0.001$	$\Delta_\mu = 0.01$	$\Delta_\mu = 0.1$	$\Delta_\mu = 1$
10	0.0532	0.0616	0.7594	1
20	0.04876	0.3436	1	1
30	0.0738	0.9908	1	1
60	0.6956	1	1	1

(c) Signal-to-noise ratio for the larger-the-better case

Sample size	$\Delta_\mu = 0.001$	$\Delta_\mu = 0.01$	$\Delta_\mu = 0.1$	$\Delta_\mu = 1$
10	0.0500	0.0581	0.7613	1
20	0.0517	0.3437	1	1
30	0.0732	0.9912	1	1
60	0.6922	1.0000	1	1

3.6. Monte Carlo Study of Properties of Proposed Tests

Table 3.4 contains the estimated powers, obtained in changing the population variances and maintaining population means at a constant value. In this case,

$$\mu_X = \mu_Y, \sigma_Y = \sigma_X + \Delta_\sigma.$$

The row entries represent the proportion of times H_0 was rejected at $\alpha = 0.05$ under H_1 .

Table 3.4: Estimated powers of t test for various sample sizes and various increments, obtained in changing the population variances and maintaining population means at constant values.

(a) Signal-to-noise ratio for the nominal-the-best case

Sample size	$\Delta_\sigma = 0.001$	$\Delta_\sigma = 0.01$	$\Delta_\sigma = 0.1$	$\Delta_\sigma = 1$
10	0.0499	0.0841	0.9997	1
20	0.0602	0.9000	1	1
30	0.1471	1	1	1
60	0.9995	1	1	1

(b) Signal-to-noise ratio for the smaller-the-better case

Sample size	$\Delta_\sigma = 0.001$	$\Delta_\sigma = 0.01$	$\Delta_\sigma = 0.1$	$\Delta_\sigma = 1$
10	0.0524	0.0531	0.0561	0.0845
20	0.0547	0.0548	0.1298	0.8457
30	0.0515	0.0563	0.6348	1
60	0.0531	0.2881	1	1

(c) Signal-to-noise ratio for the larger-the-better case

Sample size	$\Delta_\sigma = 0.001$	$\Delta_\sigma = 0.01$	$\Delta_\sigma = 0.1$	$\Delta_\sigma = 1$
10	0.0497	0.0500	0.0504	0.0820
20	0.0497	0.0498	0.1288	0.8486
30	0.0500	0.0557	0.6311	1
60	0.0522	0.2831	1	1

Results in Tables 3.2, 3.3 and 3.4 show that the estimated powers of t test increase as the increments increase. Effects of sample sizes to the estimated powers of t test are apparent. For the same value of increment in the population parameters, the proposed test detects a significance difference between two values of signal-to-noise ratios, with high power, if the corresponding sample size is also high.

3.7. Real examples

3.7 Real examples

3.7.1 Example 1 (Signal-to-noise ratio for the nominal-the-best case)

We revisit the problem of a robust design conducted on a chemical process (Lawson (2010)). The target value is set at $T = 7.6$ for convenience. Complete data are in Table A.3. The data obtained for the first two runs of the experiment are in Table 3.5.

Table 3.5: Mean and variance values for the first two runs of the chemical process.

Experiment	x_1	x_2	x_3	z_1	1	-1	-1	1	Mean	Variance
				z_2	-1	-1	1	1		
1	-1	-1	0	57.81	37.29	42.87	47.07	46.26	75.34	
2	1	-1	0	24.89	4.35	8.23	14.69	13.04	80.60	

We compare the signal-to-noise ratio of the first two experiments. Conducting the required calculations leads to the results summarized in Table 3.6.

Table 3.6: Results for the test $H_0 : SNR_{T_1} = SNR_{T_2}$.

\widehat{SNR}_{T_1}	\widehat{SNR}_{T_2}	$\widehat{\sigma}_{\widehat{SNR}_{T_1} - \widehat{SNR}_{T_2}}$	t	$t_{\alpha/2, \nu}$
14.53	3.24	2.67	4.23	2.45

As $t = 4.23 > t_{\alpha/2, \nu} = 2.45$, we conclude that SNR_{T_1} and SNR_{T_2} are statistically different at the level of significance $\alpha = 0.05$.

3.7.2 Example 2 (Signal-to-noise ratio for the smaller-the-better case)

We revisit the problem of a robust design conducted on a chemical process (Lawson (2010)) and consider its original version where the objective was to minimize the proportion of impurities in the final product. Complete data are in Table A.4. Table 3.7 presents the data obtained for the first two runs of the experiment.

Table 3.7: Proportion of impurities for the first two runs of the chemical process.

Experiment	x_1	x_2	x_3	z_1	1	-1	-1	1
				z_2	-1	-1	1	1
1	-1	-1	0	57.81	37.29	42.87	47.07	
2	1	-1	0	24.89	4.35	8.23	14.69	

3.7. Real examples

After calculations, we get the following results.

Table 3.8: Results for the test $H_0 : SNR_{S_1} = SNR_{S_2}$.

\widehat{SNR}_{S_1}	\widehat{SNR}_{S_2}	$\widehat{\sigma}_{\widehat{SNR}_{S_1} - \widehat{SNR}_{S_2}}$	t	$t_{\alpha/2, \nu}$
-33.45	-23.99	13.75	-0.688	2.45

As $|t| = 0.688 < t_{\alpha/2, \nu} = 2.45$, one concludes that SNR_{S_1} and SNR_{S_2} are not statistically different at the level of significance $\alpha = 0.05$.

3.7.3 Example 3 (Signal-to-noise for the larger-the-better case)

We use the data of a designed experiment for the optimization of transesterification of rapeseed oil (Kim and Yim (2010)). The yields of rapeseed methyl ester, as biodiesel, prepared under two sets of experimental conditions are shown in Table A.5, and the corresponding reciprocal observations are in Table A.6. All experiments were performed with three repetitions, under the same experimental conditions (e.g., molar ratio of alcohol to oil, catalyst type, catalyst concentration and reaction temperature). Table 3.9a and Table 3.9b give the data for the first two runs and the corresponding reciprocals, respectively.

Table 3.9: Yields of rapeseed methyl ester.

(a) Data for the first two runs of Yields of rapeseed methyl ester

Run	Sample 1	Sample 2	Sample 3
1	52.1	76.7	84.4
2	72.5	91.4	90.3

(Kim and Yim (2010))

(b) Reciprocal values

Run	Sample 1	Sample 2	Sample 3
1	0.0192	0.0130	0.0118
2	0.0138	0.0109	0.0111

After calculations, we get the following results.

Table 3.10: Results for the test $H_0 : SNR_{L_1} = SNR_{L_2}$

\widehat{SNR}_{L_1}	\widehat{SNR}_{L_2}	$\widehat{\sigma}_{\widehat{SNR}_{L_1} - \widehat{SNR}_{L_2}}$	t	$t_{\alpha/2, \nu}$
36.37	38.39	29.69	-0.068	2.78

3.8. Conclusions

As $|t| = 0.068 < t_{\alpha/2, \nu} = 2.78$, one concludes that SNR_{L_1} and SNR_{L_2} are not statistically different at the level of significance $\alpha = 0.05$.

3.8 Conclusions

A brief review of the combined array design is provided and the related metric known as signal-to-noise ratio is presented. Based on multivariate Delta theorem, the asymptotic distributions of signal-to-noise ratios are determined. Considering the main classes of possible responses, we propose a statistical test for pairwise comparisons of treatments with regard to the signal-to-noise ratio. The correction to this pairwise comparison can be done using the Bonferroni inequality as stated by [Chang \(2008\)](#). The correction consists in applying the adjusted level of significance and adjusted p -value. Illustrative examples based on simulation and on real data are presented. The values of the estimated test sizes are displayed by [Table 3.1](#). [Tables 3.2, 3.3, and 3.4](#) display the values of the estimated test powers, according to the three scenarios presented in the paragraph on Procedure for Monte Carlo simulation. The statistical test we propose preserves the test size when simulations are conducted under H_0 and has excellent powers when simulations are conducted under H_1 .

Chapter 4

General conclusions

In this thesis, we provide a methodology for conducting robust designs and analyse the data obtained from the design. The general assumption in robust design is that there are two types of factors: the design factors which are controlled by the experimenter, and the noise factors which are difficult to control. As the noise factors are responsible for the variability which affects the quality characteristic of the process, the aim of robust design is to achieve insensitivity to noise factors. This consists of bringing the quality characteristic to the target, while simultaneously minimizing its variance and the cost of the process.

The experimental designs we present in this thesis are the Taguchi approach commonly known as double orthogonal array design or crossed-array design, and combined array design. The Taguchi approach consists of a double orthogonal array design, one for the design factors and another for the noise factors. The combined array design puts both types of factors in one design. This design permits the analysis of interactions between the designs and noise factors, and reduces the number of runs required to conduct an experiment. The data analysis in combined array design consists of adjusting a regression model in terms of design factors and noise factors. From this adjusted model, two response surfaces are obtained, one for the mean of the quality characteristic, and another for its variance.

As the aim of robust designs is to make the process insensitive or less sensitive to the effects of noise factors, the problem reduces to determining the optimal settings of control designs so that the effects of noise factors on the process are minimized. We solve the optimization problems that arise from this procedure using the Mean square error criterion and Desirability function.

In practice, it is common to observe a multidimensional quality characteristic. This thesis presents a desirability function based on mean square error criterion for handling multiple responses in robust design.

We illustrate the double orthogonal array design with an example. The data consists of an application of robust design conducted on a chemical process, as presented by [Lawson \(1990\)](#)

4. General conclusions

and Lawson (2010). The objective of the application is to conduct a designed experiment in order to obtain the operating conditions that improve the process. The improvement consists of the reduction of the proportion of impurities in the product, and the minimization of the cost. We use the quality loss function in order to evaluate the cost caused by the deviation from the target value of the quality characteristic.

To illustrate the combined array design, we use the desirability function based on mean square error criterion. We analyse the data presented by Romano et al. (2004) and K oksoy (2006). The problem is a robust design conducted on the elastic element of a force transducer. The main objective of the design is to determine the optimal conditions for lozenge angle, bore diameter and half-length of the vertical segment, here stated as control factors, so that the effects of the deviation of the lozenge angle from its target value, and the deviation of the bore diameter from its nominal value, both considered as noise factors, are insensitive to two response variables, these are, the non-linearity and hysteresis. We seek to achieve a target value of 1 for the non-linearity and minimize the hysteresis.

One of the contributions of Taguchi to robust design is the use of the metric known as Signal-to-Noise Ratio. This metric takes into account the mean of the quality characteristic and its variance. Multiple comparison is one of the topics with utmost importance in any designed experiment. In this thesis, we propose a statistical test for pairwise comparisons of experiments. This statistical test exploits the asymptotic normality of the estimates of signal-to-noise ratios, and uses the Central limit theorem to derive its asymptotic distribution. Simulated examples and real examples are given. Simulated examples show that the test size is close to the significance level (α) when simulations are conducted under H_0 , this is, considering equal population parameters. Moreover, simulations under H_1 also show that the proposed statistical tests have good powers.

This thesis is a contribution to robust designed experiments, but still, there is a room for further improvement. Future research may concern multiple comparisons of treatments based on signal-to-noise ratio. Moreover, even though robust designs have been extensively applied in various areas, analytical studies of statistical properties of parameter estimates are not deeply studied. Hence, this is another interesting topic for our future research. It is also interesting to include, in our future research, the analysis of categorical response variables, and the concept of censored observations.

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Appendix A

Data

A.1 Data for the chemical process problem

Table A.1: Data for the chemical process problem

(a) Coded and real levels

	Levels		
Design factors	-1	0	1
x_1	180	210	240
x_2	25	30	35
x_3	12	15	18
Noise factors	-1	1	
z_1	10	20	
z_2	30	40	

(b) Experimental results of the chemical process

Experiment	x_1	x_2	x_3	z_1	1	-1	-1	1	Mean	$\ln(s^2)$
				z_2	-1	-1	1	1		
1	-1	-1	0	57.81	37.29	42.87	47.07	46.26	4.32	
2	1	-1	0	24.89	4.35	8.23	14.69	13.04	4.39	
3	-1	1	0	13.21	9.51	10.10	11.19	11.00	0.97	
4	1	1	0	13.29	9.15	10.30	11.23	10.99	1.17	
5	-1	0	-1	27.71	20.24	22.28	24.23	23.62	2.31	
6	1	0	-1	11.40	4.48	5.44	8.23	7.39	2.27	
7	-1	0	1	30.65	18.40	20.24	24.45	23.44	3.39	
8	1	0	1	14.94	2.29	4.30	8.49	7.51	3.44	
9	0	-1	-1	42.68	22.42	21.64	30.30	29.26	4.56	
10	0	1	-1	13.56	10.08	9.85	11.38	11.22	1.06	
11	0	-1	1	50.60	13.19	18.84	30.97	28.40	5.61	
12	0	1	1	15.21	7.44	9.78	11.82	11.06	2.38	
13	0	0	0	19.62	12.29	13.14	14.54	14.90	2.38	
14	0	0	0	20.60	11.49	12.06	13.49	14.41	2.88	
15	0	0	0	20.15	12.20	14.06	13.89	15.08	2.50	

Source: [Lawson \(2010\)](#)

A.2. Data for the elastic element problem

A.2 Data for the elastic element problem

Table A.2: Data for the elastic element problem

(a) Coded and real levels

	Levels		
Design factors	-1	0	1
x_1	15	30	45
x_2	8	11	14
x_3	7	9	11
Noise factors	-1	0	1
z_1	-1.5	0	1.5
z_2	-0.25	0	0.25

(b) Experimental results of the elastic element problem

Treatment	x_1	x_2	x_3	z_1	z_2	y_1	y_2
1	-1	-1	-1	-1	1	1.810	1.10
2	-1	-1	-1	1	-1	1.690	1.11
3	-1	-1	1	-1	-1	1.900	1.07
4	-1	-1	1	1	1	1.780	1.07
5	-1	1	-1	-1	-1	1.800	1.47
6	-1	1	-1	1	1	1.630	1.18
7	-1	1	1	-1	1	1.920	1.41
8	-1	1	1	1	-1	1.780	1.58
9	1	-1	-1	-1	-1	1.360	1.57
10	1	-1	-1	1	1	1.220	2.03
11	1	-1	1	-1	1	1.480	1.38
12	1	-1	1	1	-1	1.440	1.68
13	1	1	-1	-1	1	0.693	3.37
14	1	1	-1	1	-1	0.616	3.75
15	1	1	1	-1	-1	0.950	2.81
16	1	1	1	1	1	0.817	2.83
17	-1	0	0	0	0	1.790	1.24
18	1	0	0	0	0	1.030	2.46
19	0	-1	0	0	0	1.530	1.23
20	0	1	0	0	0	1.220	1.73
21	0	0	-1	0	0	1.300	1.63
22	0	0	1	0	0	1.440	1.67
23	0	0	0	0	0	1.380	1.73
24	0	0	0	0	0	1.390	1.74
25	0	0	0	0	0	1.400	1.74

Source: [Romano et al. \(2004\)](#)

A.3. Chemical process problem and values of $\widehat{\text{SNR}}_T$

A.3 Chemical process problem and values of $\widehat{\text{SNR}}_T$

Table A.3: Chemical process problem and values of $\widehat{\text{SNR}}_T$

Experiment	x_1	x_2	x_3	z_1	1	-1	-1	1	Mean	Variance	$\widehat{\text{SNR}}_T$
				z_2	-1	-1	1	1			
1	-1	-1	0	57.81	37.29	42.87	47.07	46.26	75.34	14.53	
2	1	-1	0	24.89	4.35	8.23	14.69	13.04	80.60	3.24	
3	-1	1	0	13.21	9.51	10.10	11.19	11.00	2.65	16.60	
4	1	1	0	13.29	9.15	10.30	11.23	10.99	3.23	15.73	
5	-1	0	-1	27.71	20.24	22.28	24.23	23.62	10.11	17.42	
6	1	0	-1	11.40	4.48	5.44	8.23	7.39	9.69	7.51	
7	-1	0	1	30.65	18.40	20.24	24.45	23.44	29.55	12.69	
8	1	0	1	14.94	2.29	4.30	8.49	7.51	31.24	2.57	
9	0	-1	-1	42.68	22.42	21.64	30.30	29.26	95.34	9.53	
10	0	1	-1	13.56	10.08	9.85	11.38	11.22	2.89	16.39	
11	0	-1	1	50.60	13.19	18.84	30.97	28.40	274.06	4.69	
12	0	1	1	15.21	7.44	9.78	11.82	11.06	10.85	10.52	
13	0	0	0	19.62	12.29	13.14	14.54	14.90	10.77	13.14	
14	0	0	0	20.60	11.49	12.06	13.49	14.41	17.74	10.68	
15	0	0	0	20.15	12.20	14.06	13.89	15.08	12.15	12.72	

A.4 Chemical process problem and values of $\widehat{\text{SNR}}_S$

Table A.4: Chemical process problem and values of $\widehat{\text{SNR}}_S$

Experiment	x_1	x_2	x_3	z_1	1	-1	-1	1	$\widehat{\text{SNR}}_S$
				z_2	-1	-1	1	1	
1	-1	-1	0	57.81	37.29	42.87	47.07	-33.45	
2	1	-1	0	24.89	4.35	8.23	14.69	-23.99	
3	-1	1	0	13.21	9.51	10.10	11.19	-20.92	
4	1	1	0	13.29	9.15	10.30	11.23	-20.96	
5	-1	0	-1	27.71	20.24	22.28	24.23	-27.54	
6	1	0	-1	11.40	4.48	5.44	8.23	-18.08	
7	-1	0	1	30.65	18.40	20.24	24.45	-27.62	
8	1	0	1	14.94	2.29	4.30	8.49	-19.42	
9	0	-1	-1	42.68	22.42	21.64	30.30	-29.78	
10	0	1	-1	13.56	10.08	9.85	11.38	-21.10	
11	0	-1	1	50.60	13.19	18.84	30.97	-30.34	
12	0	1	1	15.21	7.44	9.78	11.82	-21.25	
13	0	0	0	19.62	12.29	13.14	14.54	-23.67	
14	0	0	0	20.60	11.49	12.06	13.49	-23.53	
15	0	0	0	20.15	12.20	14.06	13.89	-23.79	

A.5. Yields of rapeseed methyl ester

A.5 Yields of rapeseed methyl ester

Table A.5: Yields of rapeseed methyl ester

Run	Sample 1	Sample 2	Sample 3
1	52.1	76.7	84.4
2	72.5	91.4	90.3
3	74.5	94.6	91.5
4	71.0	87.2	85.5
5	69.0	90.2	91.5
6	51.5	83.8	78.2
7	69.7	88.2	85.4
8	63.5	81.9	80.5
9	65.1	79.8	84.4

A.6 Reciprocal values of Yields of rapeseed methyl ester and values of $\widehat{\text{SNR}}_L$

Table A.6: Reciprocal values of Yields of rapeseed methyl ester and values of $\widehat{\text{SNR}}_L$

Run	Sample 1	Sample 2	Sample 3	$\widehat{\text{SNR}}_L$
1	0.0192	0.0130	0.0118	36.37
2	0.0138	0.0109	0.0111	38.39
3	0.0134	0.0106	0.0109	38.61
4	0.0141	0.0115	0.0117	38.05
5	0.0145	0.0111	0.0109	38.18
6	0.0194	0.0119	0.0128	36.33
7	0.0143	0.0113	0.0117	38.03
8	0.0157	0.0122	0.0124	37.34
9	0.0154	0.0125	0.0118	37.48

Appendix B

Calculations

B.1 Estimated mean models: Chapter 2, Example 3

$$\begin{aligned} & \widehat{E}_{\mathbf{z}}(y^{(1)}(\mathbf{x}, \mathbf{z})) \\ &= \widehat{E}_{\mathbf{z}}(1.377 - 0.361x_1 - 0.155x_2 + 0.077x_3 + 0.042x_1^2 + 0.007x_2^2 + 0.002x_3^2 - 0.148x_1x_2 \\ &\quad + 0.022x_1x_3 + 0.013x_2x_3 - 0.059z_1 - 0.012z_2 + 0.010x_1z_1 - 0.008x_1z_2 - 0.006x_2z_1 \\ &\quad + 0.001x_2z_2 + 0.005x_3z_1 + 0.003x_3z_2) \\ &= \widehat{E}_{\mathbf{z}}(1.377 - 0.361x_1 - 0.155x_2 + 0.077x_3 + 0.042x_1^2 + 0.007x_2^2 + 0.002x_3^2 - 0.148x_1x_2 \\ &\quad + 0.022x_1x_3 + 0.013x_2x_3) + \widehat{E}_{\mathbf{z}}(-0.059z_1 - 0.012z_2 + 0.010x_1z_1 - 0.008x_1z_2 \\ &\quad - 0.006x_2z_1 + 0.001x_2z_2 + 0.005x_3z_1 + 0.003x_3z_2) \\ &= \widehat{E}_{\mathbf{z}}(1.377 - 0.361x_1 - 0.155x_2 + 0.077x_3 + 0.042x_1^2 + 0.007x_2^2 + 0.002x_3^2 - 0.148x_1x_2 \\ &\quad + 0.022x_1x_3 + 0.013x_2x_3) \\ &= 1.377 - 0.361x_1 - 0.155x_2 + 0.077x_3 + 0.042x_1^2 + 0.007x_2^2 + 0.002x_3^2 - 0.148x_1x_2 \\ &\quad + 0.022x_1x_3 + 0.013x_2x_3. \end{aligned}$$

B.2. Estimated variance models: Chapter 2, Example 3

$$\begin{aligned}
& \widehat{E}_{\mathbf{z}}(y^{(2)}(\mathbf{x}, \mathbf{z})) \\
&= \widehat{E}_{\mathbf{z}}(1.660 + 0.592x_1 + 0.438x_2 - 0.095x_3 + 0.247x_1^2 - 0.123x_2^2 + 0.047x_3^2 + 0.301x_1x_2 \\
&\quad - 0.143x_1x_3 - 0.033x_2x_3 + 0.066z_1 - 0.042z_2 + 0.079x_1z_1 + 0.017x_1z_2 - 0.031x_2z_1 \\
&\quad - 0.061x_2z_2 - 0.004x_3z_1 - 0.014x_3z_2) \\
&= \widehat{E}_{\mathbf{z}}(1.660 + 0.592x_1 + 0.438x_2 - 0.095x_3 + 0.247x_1^2 - 0.123x_2^2 + 0.047x_3^2 + 0.301x_1x_2 \\
&\quad - 0.143x_1x_3 - 0.033x_2x_3) + \widehat{E}_{\mathbf{z}}(0.066z_1 - 0.042z_2 + 0.079x_1z_1 + 0.017x_1z_2 - 0.031x_2z_1 \\
&\quad - 0.061x_2z_2 - 0.004x_3z_1 - 0.014x_3z_2) \\
&= \widehat{E}_{\mathbf{z}}(1.660 + 0.592x_1 + 0.438x_2 - 0.095x_3 + 0.247x_1^2 - 0.123x_2^2 + 0.047x_3^2 + 0.301x_1x_2 \\
&\quad - 0.143x_1x_3 - 0.033x_2x_3) \\
&= 1.660 + 0.592x_1 + 0.438x_2 - 0.095x_3 + 0.247x_1^2 - 0.123x_2^2 + 0.047x_3^2 + 0.301x_1x_2 \\
&\quad - 0.143x_1x_3 - 0.033x_2x_3.
\end{aligned}$$

B.2 Estimated variance models: Chapter 2, Example 3

$$\begin{aligned}
& \widehat{Var}_{\mathbf{z}}(y^{(1)}(\mathbf{x}, \mathbf{z})) \\
&= \left(\widehat{\boldsymbol{\gamma}}^{(1)T} + \mathbf{x}^T \widehat{\boldsymbol{\Delta}}^{(1)} \right) \boldsymbol{\Omega} \left(\widehat{\boldsymbol{\gamma}}^{(1)} + \widehat{\boldsymbol{\Delta}}^{(1)T} \mathbf{x} \right) + \widehat{\sigma}_{\epsilon^{(1)}}^2 \\
&= \left[\begin{pmatrix} -0.059 & -0.012 \end{pmatrix} + \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 0.010 & -0.008 \\ -0.006 & 0.001 \\ 0.005 & 0.003 \end{pmatrix} \right] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&\quad \left[\begin{pmatrix} -0.059 \\ -0.012 \end{pmatrix} + \begin{pmatrix} 0.010 & -0.006 & 0.005 \\ -0.008 & 0.001 & 0.003 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right] + 0.0003 \\
&= 3.625 \times 10^{-3} - 9.88 \times 10^{-4}x_1 + 6.84 \times 10^{-4}x_2 - 6.62 \times 10^{-4}x_3 + 1.64 \times 10^{-4}x_1^2 \\
&\quad + 3.7 \times 10^{-5}x_2^2 + 3.4 \times 10^{-5}x_3^2 - 1.36 \times 10^{-4}x_1x_2 + 5.2 \times 10^{-5}x_1x_3 \\
&\quad - 5.4 \times 10^{-5}x_2x_3 + 0.0003 \\
&= 3.925 \times 10^{-3} - 9.88 \times 10^{-4}x_1 + 6.84 \times 10^{-4}x_2 - 6.62 \times 10^{-4}x_3 + 1.64 \times 10^{-4}x_1^2 \\
&\quad + 3.7 \times 10^{-5}x_2^2 + 3.4 \times 10^{-5}x_3^2 - 1.36 \times 10^{-4}x_1x_2 + 5.2 \times 10^{-5}x_1x_3 \\
&\quad - 5.4 \times 10^{-5}x_2x_3.
\end{aligned}$$

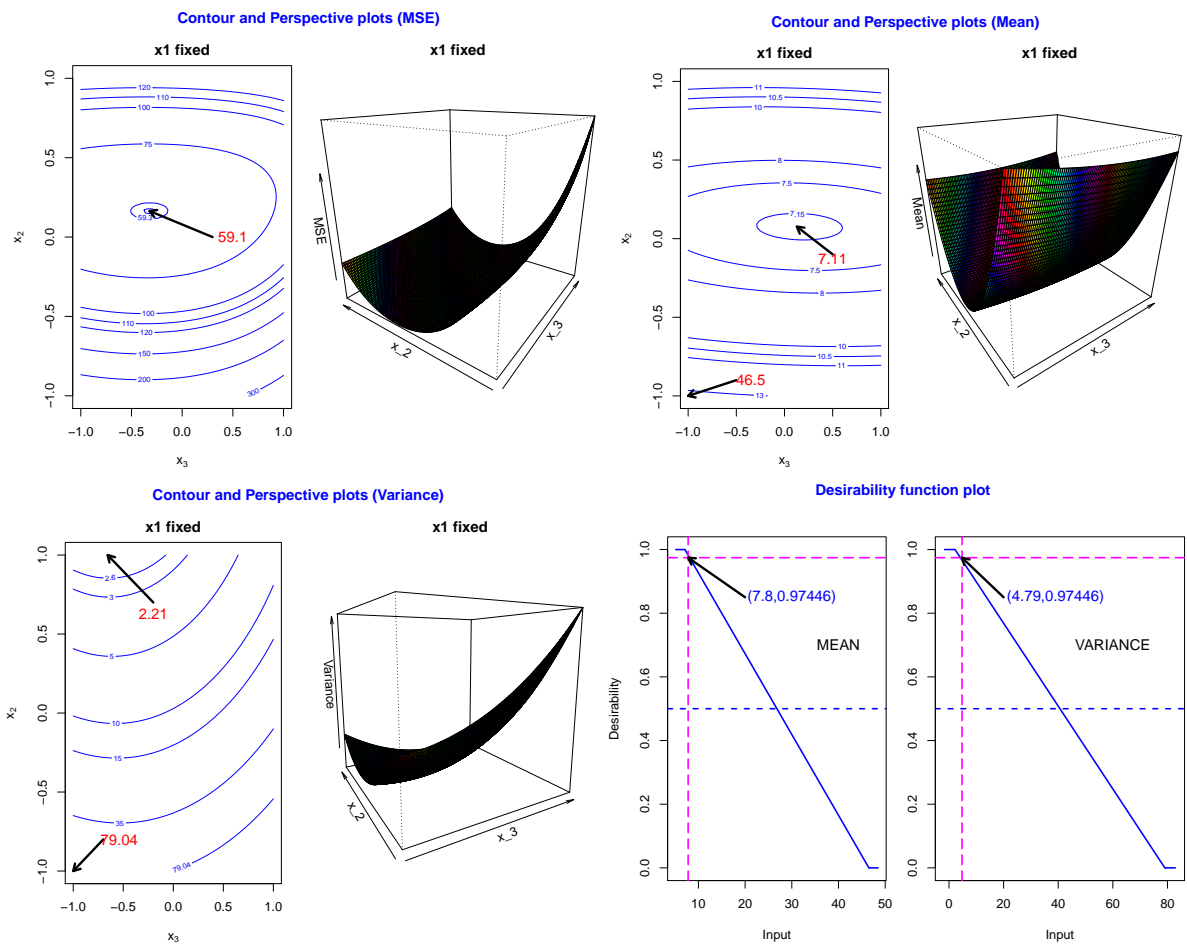
B.2. Estimated variance models: Chapter 2, Example 3

$$\begin{aligned}
\widehat{Var}_z(y^{(2)}(\mathbf{x}, \mathbf{z})) &= \left(\widehat{\boldsymbol{\gamma}}^{(2)T} + \mathbf{x}^T \widehat{\boldsymbol{\Delta}}^{(2)} \right) \boldsymbol{\Omega} \left(\widehat{\boldsymbol{\gamma}}^{(2)} + \widehat{\boldsymbol{\Delta}}^{(2)T} \mathbf{x} \right) + \widehat{\sigma}_{\epsilon^{(2)}}^2 \\
&= \left[\begin{array}{cc} 0.066 & -0.042 \end{array} \right] + \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 0.079 & 0.017 \\ -0.031 & -0.061 \\ -0.004 & -0.014 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&\quad \left[\begin{pmatrix} 0.066 \\ -0.042 \end{pmatrix} + \begin{pmatrix} 0.079 & -0.031 & -0.004 \\ 0.017 & -0.061 & -0.014 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right] + 0.037 \\
&= 0.00612 + 0.009x_1 + 1.032 \times 10^{-3}x_2 + 6.48 \times 10^{-4}x_3 + 0.00653x_1^2 \\
&\quad + 4.682 \times 10^{-3}x_2^2 + 2.12 \times 10^{-4}x_3^2 - 6.972 \times 10^{-3}x_1x_2 - 1.108 \times 10^{-4}x_1x_3 \\
&\quad + 1.956x_2x_3 + 0.037 \\
&= 0.04312 + 0.009x_1 + 1.032 \times 10^{-3}x_2 + 6.48 \times 10^{-4}x_3 + 0.00653x_1^2 \\
&\quad + 4.682 \times 10^{-3}x_2^2 + 2.12 \times 10^{-4}x_3^2 - 6.972 \times 10^{-3}x_1x_2 - 1.108 \times 10^{-4}x_1x_3 \\
&\quad + 1.956x_2x_3.
\end{aligned}$$

Appendix C

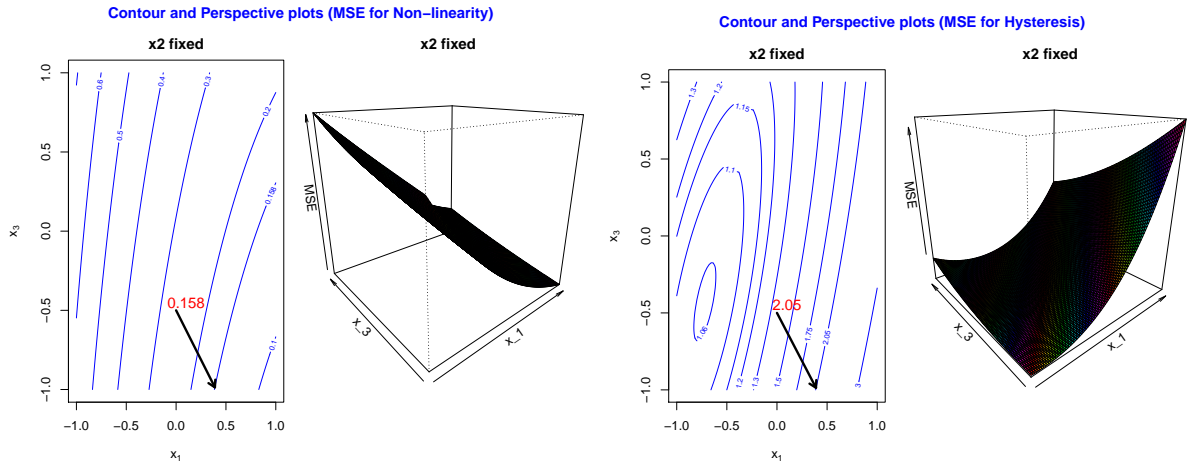
Graphs

C.1 Graphs for the chemical process problem

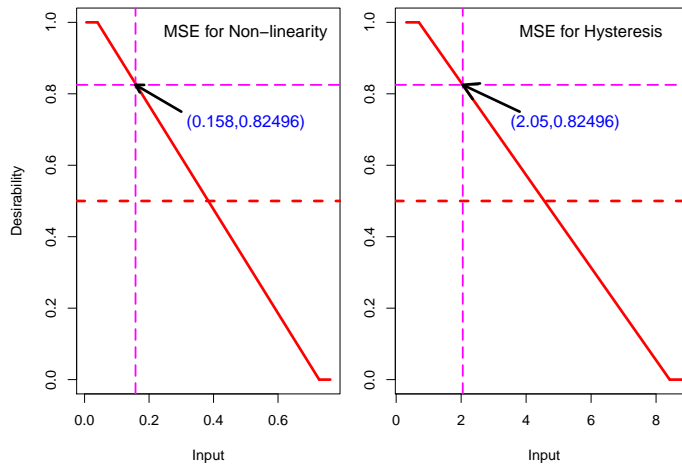


C. Graphs

C.2 Graphs for the elastic element of force transducer problem

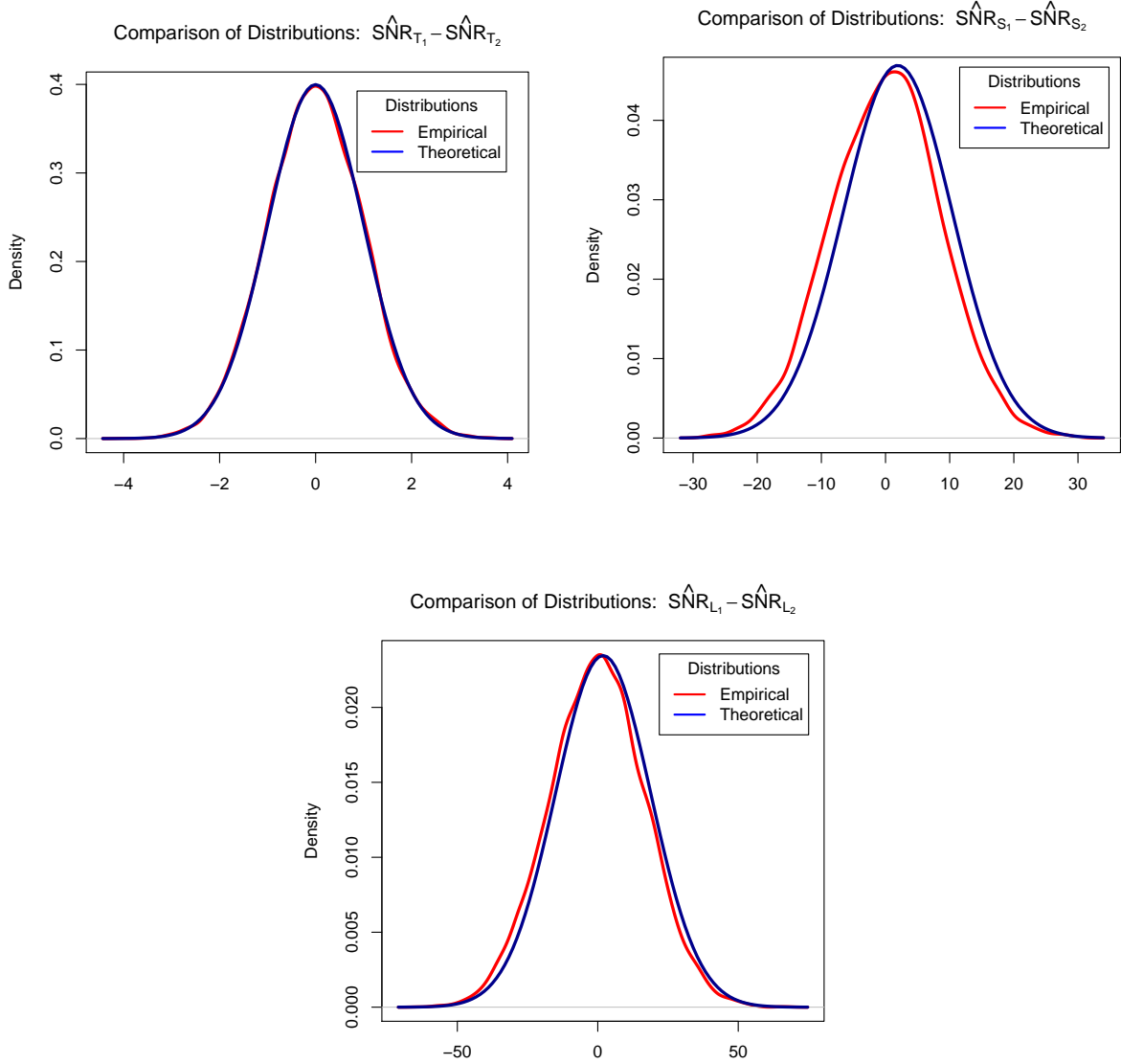


Plot for the Desirability Function



C. Graphs

C.3 Distribution of $\widehat{SNR}_1 - \widehat{SNR}_2$



Appendix D

R Codes

D.1 R Code for MSE Analysis (Chemical process problem)

```
##### PART 1 #####
#####
# This R Code determines x_opt and the corresponding MSE.
# The proportion of impurities and its variance are obtained
# by introducing x_opt in the adjusted models for Y_Mean and Y_Var.
#####
rm(list=ls())
library("nloptr")
x0 <- c( 0, 0, 0 )
fn <- function(x) (14.80 - 8.17 * x[1] - 9.09 * x[2] - 0.14 * x[3]
  + 0.52 * x[1]^2 + 8.30 * x[1] * x[2] + 0.07 * x[1] * x[3]
  + 5.01 * x[2]^2 + 0.18 * x[2] * x[3] + 0.18 * x[3]^2 )^2
  +(exp(2.59 + 0.03 * x[1] - 1.66 * x[2] + 0.58 * x[3]
  - 0.21 * x[1]^2 + 0.03 * x[1] * x[2] + 0.02 * x[1] * x[3]
  + 0.34 * x[2]^2 + 0.07 * x[2] * x[3] + 0.48 * x[3]^2))
hin <- function(x) c( x[1] + 1, x[2] + 1, x[3] + 1, -x[1] + 1, -x[2] + 1,
  -x[3] + 1)
gr <- function(x) nl.grad(x, fn)
hinjac <- function(x) nl.jacobian(x, hin)
auglag(x0, fn, gr = NULL, hin = hin, localsolver = "SLSQP")
##### x_opt #####
x1 = 1.00000000
x2 = 0.1640518
x3 = -0.3217378
#####
MeanPred <- (14.80 - 8.17 * x1 - 9.09 * x2 - 0.14 * x3 + 0.52 * x1^2
  + 8.30 * x1 * x2 + 0.07 * x1 * x3 + 5.01 * x2^2
  + 0.18 * x2 * x3 + 0.18 * x3^2)
VariancePred <- (exp(2.59 + 0.03 * x1 - 1.66 * x2 + 0.58 * x3 - 0.21 * x1^2
  + 0.03 * x1 * x2 + 0.02 * x1 * x3 + 0.34 * x2^2 + 0.07 * x2 * x3
  + 0.48 * x3^2))
MSE_DOAD <- ((14.80 - 8.17 * x1 - 9.09 * x2 - 0.14 * x3 + 0.52 * x1^2
```

D.1. R Code for MSE Analysis (Chemical process problem)

```
+ 8.30 * x1 * x2 + 0.07 * x1 * x3 + 5.01 * x2^2 + 0.18 * x2 * x3
  + 0.18 * x3^2)^2 + (exp(2.59 + 0.03 * x1 - 1.66 * x2 + 0.58 * x3
- 0.21 * x1^2 + 0.03 * x1 * x2 + 0.02 * x1 * x3 + 0.34 * x2^2
  + 0.07 * x2 * x3 + 0.48 * x3^2)))
print(MeanPred,      digits = 3)
print(VariancePred, digits = 3)
print(MSE_DLOAD,    digits = 3)
##### END PART 1 #####

##### PART 2 #####
#####
# This R Code graphs the Contour plot and Perspective plot
# for the proportion of impurities (Mean), its variance and the
# corresponding MSE
#####

# Contour and perspective plots- Mean #####
##### x_1 fixed #####
par(oma = c(0, 0, 2, 0))
par(mfrow = c(1, 2))
par(mar = c(4.1, 4.1, 2.1, 1.1))
x_1 <- 1
x_2 <- seq(-1, 1, len = 100)
x_3 <- seq(-1, 1, len = 100)
f <- outer(x_3, x_2, function(x_3,x_2) 14.80 - 8.17 * x_1 - 9.09 * x_2
- 0.14 * x_3 + 0.52 * x_1^2 + 8.30 * x_1 * x_2 + 0.07 * x_1 * x_3 + 5.01 * x_2^2
+ 0.18 * x_2 * x_3 + 0.18 * x_3^2)
contour(x_3, x_2, f, levels = c(13, 11, 10.5, 10, 8, 7.5, 7.15, 7.11),
xlab = expression(x[3]), ylab = expression(x[2]), main = "x1 fixed", col = "blue")
arrows(0.5, -0.10, 0.12839, 0.07654, length = 0.1, pch = 2, lwd=3)
text(0.5, -0.13, "7.11", cex = 1.2, col = "red")
arrows(-0.5, -0.9, -1,-1, length=0.1, pch = 2, lwd=3)
text(-0.35, -0.9, "46.5", cex=1.2, col = "red")

par(mar = c(3.1, 0.1, 2.1, 0.1))
y <- function(x_3, x_2){ 14.80 - 8.17 * x_1 - 9.09 * x_2 - 0.14 * x_3
+ 0.52 * x_1^2 + 8.30 * x_1 * x_2 + 0.07 * x_1 * x_3
+ 5.01 * x_2^2 + 0.18 * x_2 * x_3 + 0.18 * x_3^2}
f <- outer(x_3, x_2, y)
persp(x_3, x_2, f, theta = -35, phi = 25, col = rainbow(50),
xlab = expression(x_3), ylab = expression(x_2),
zlab =expression(Mean), main = "x1 fixed")
title(main = "Contour and Perspective plots (Mean)", sub = "Nothing",
col.main="blue", outer = T)
#####
# Contour and perspective plots- Variance #####
##### x_1 fixed #####
par(oma = c(0, 0, 2, 0))
par(mfrow = c(1, 2))
par(mar = c(4.1, 4.1, 2.1, 1.1))
x_1 <- -1
x_2 <- seq(-1, 1, len = 100)
x_3 <- seq(-1, 1, len = 100)
```

D.1. R Code for MSE Analysis (Chemical process problem)

```
f <- outer(x_3, x_2, function(x_3, x_2) exp(2.59 + 0.03 * x_1 - 1.66 * x_2
+ 0.58 * x_3 - 0.21 * x_1^2 + 0.03 * x_1 * x_2 + 0.02 * x_1 * x_3
+ 0.34 * x_2^2 + 0.07 * x_2 * x_3 + 0.48 * x_3^2))
contour(x_3, x_2, f, levels = c(79.04, 35, 15, 10, 5, 3, 2.6),
xlab = expression(x[3]), ylab = expression(x[2]), main= "x1 fixed",
col = "blue")
arrows(-0.2, 0.7, -0.65625,1, length = 0.1, pch = 2, lwd = 3)
text(-0.2, 0.63, "2.21", cex = 1.2, col = "red")
arrows(-0.7, -0.8, -1,-1, length = 0.1, pch = 2, lwd=3)
text(-0.54, -0.8, "79.04", cex = 1.2, col = "red")

par(mar=c(3.1,1.9,2.1,0.1))
y<-function(x_3,x_2){ 13.55 + 0.89 * x_1 - 63.22 * x_2 + 28.46 * x_3
- 24.37 * x_1^2 - 1.17 * x_1 * x_2 + 0.53 * x_1 * x_3 + 51.27 * x_2^2
- 42.69 * x_2 * x_3 + 30.96 * x_3^2}
f <- outer(x_3, x_2 ,y)
persp(x_3, x_2, f, theta = -30, phi = 10,col = rainbow(50),
xlab = expression(x_3), ylab = expression(x_2), zlab = expression(Variance),
main = "x1 fixed")
title(main = "Contour and Perspective plots (Variance)", sub = "Nothing",
col.main = "blue", outer = T)
#####
#Contour and perspective plots- MSE #####
##### x_1 fixed #####
par(oma = c(0, 0, 2, 0))
par(mfrow = c(1, 2))
par(mar = c(4.1, 4.1, 2.1, 1.1))
x_1 <- 1
x_2 <- seq(-1, 1,len = 100)
x_3 <- seq(-1, 1,len = 100)
f<-outer(x_3, x_2, function(x_3, x_2) (14.80 - 8.17 * x_1 - 9.09 * x_2
- 0.14 * x_3 + 0.52 * x_1^2 + 8.30 * x_1 * x_2 + 0.07 * x_1 * x_3
+ 5.01 * x_2^2 + 0.18 * x_2 * x_3 + 0.18 * x_3^2)^2
+ (exp(2.59 + 0.03 * x_1 - 1.66 * x_2 + 0.58 * x_3 - 0.21 * x_1^2
+ 0.03 * x_1 * x_2 + 0.02 * x_1 * x_3 + 0.34 * x_2^2 + 0.07 * x_2 * x_3
+ 0.48 * x_3^2)))
contour(x_3, x_2, f, levels = c(300, 200, 150, 120, 110, 100, 75, 59.3, 59.1),
xlab = expression(x[3]), ylab = expression(x[2]), main = "x1 fixed",
col = "blue")
arrows(0.3, 0, -0.32174, 0.16405, length = 0.1, pch = 2, lwd = 3)
text(0.5, 0, "59.1", cex = 1.2, col = "red")

par(mar = c(3.1, 0.1, 2.1, 0.1))
y <- function(x_3, x_2){ (14.80 - 8.17 * x_1 - 9.09 * x_2 - 0.14 * x_3
+ 0.52 * x_1^2 + 8.30 * x_1 * x_2 + 0.07 * x_1 * x_3 + 5.01 * x_2^2
+ 0.18 * x_2 * x_3 + 0.18 * x_3^2)^2+(exp(2.59 + 0.03 * x_1 - 1.66 * x_2
+ 0.58 * x_3 - 0.21 * x_1^2 + 0.03 * x_1 * x_2 + 0.02 * x_1 * x_3
+ 0.34 * x_2^2 + 0.07 * x_2 * x_3 + 0.48 * x_3^2))}
f <- outer(x_3, x_2, y)
persp(x_3, x_2, f, theta = -55, phi = 20,col = rainbow(50),
xlab = expression(x_3), ylab = expression(x_2),
zlab =expression(MSE), main = "x1 fixed")
title(main = "Contour and Perspective plots (MSE)", sub = "Nothing",
```

D.2. R Code for Desirability Analysis (Chemical process problem)

```
col.main="blue", outer = T)
##### END PART 2 #####
```

D.2 R Code for Desirability Analysis (Chemical process problem)

```
##### PART 1: INDIVIDUAL OPTIMIZATION #####
#####
# This R Code determines x_min and x_max in order to obtain the lower bounds and
# upper bounds to be applied in determining individual desirability functions
# for the mean model and variance model of the response variable, i.e.,
# "the proportion of impurities"
#####
##### DETERMINING "x_min" #####
##### MINIMIZING " MEAN " #####
rm(list=ls())
library("nloptr")
x0 <- c( 0, 0, 0)
fn <- function(x) (14.80 - 8.17 * x[1] - 9.09 * x[2] - 0.14 * x[3]
+ 0.52 * x[1]^2 + 8.30 * x[1] * x[2] + 0.07 * x[1] * x[3]
+ 5.01 * x[2]^2 + 0.18 * x[2] * x[3] + 0.18 * x[3]^2 )
hin <- function(x) c( x[1] + 1, x[2] + 1, x[3] + 1, -x[1] + 1, -x[2] + 1,
-x[3] + 1)
gr <- function(x) nl.grad(x, fn)
hinjac <- function(x) nl.jacobian(x, hin)
auglag(x0, fn, gr = NULL, hin = hin, localsolver = "SLSQP")
#####
##### MINIMIZING " VARIANCE " #####
rm(list=ls())
library("nloptr")
x0 <- c( 0, 0, 0 )
fn <- function(x) (exp(2.59 + 0.03 * x[1] - 1.66 * x[2] + 0.58 * x[3]
- 0.21 * x[1]^2 + 0.03 * x[1] * x[2] + 0.02 * x[1] * x[3]
+ 0.34 * x[2]^2 + 0.07 * x[2] * x[3] + 0.48 * x[3]^2) )
hin <- function(x) c( x[1] + 1, x[2] + 1, x[3] + 1, -x[1] + 1, -x[2] + 1,
-x[3] + 1)
gr <- function(x) nl.grad(x, fn)
hinjac <- function(x) nl.jacobian(x, hin)
auglag(x0, fn, gr = NULL, hin = hin, localsolver = "SLSQP")
#####
##### DETERMINING "x_max" #####
##### MAXIMIZING " MEAN " #####
rm(list=ls())
library("nloptr")
x0 <- c( -1, -1, -1)
fn <- function(x) ( -(14.80 - 8.17 * x[1] - 9.09 * x[2] - 0.14 * x[3]
+ 0.52 * x[1]^2 + 8.30 * x[1] * x[2] + 0.07 * x[1] * x[3]
+ 5.01 * x[2]^2 + 0.18 * x[2] * x[3] + 0.18 * x[3]^2) )
hin <- function(x) c( x[1] + 1, x[2] + 1, x[3] + 1, -x[1] + 1, -x[2] + 1,
-x[3] + 1)
```

D.2. R Code for Desirability Analysis (Chemical process problem)

```
gr <- function(x) nl.grad(x, fn)
hinjac <- function(x) nl.jacobian(x, hin)
auglag(x0, fn, gr = NULL, hin = hin, localsolver = "SLSQP")
#####
##### MAXIMIZING " VARIANCE " #####
rm(list=ls())
library("nloptr")
x0 <- c( -1, -1, -1)
fn <- function(x) ( -(exp(2.59 + 0.03 * x[1] - 1.66 * x[2] + 0.58 * x[3]
- 0.21 * x[1]^2 + 0.03 * x[1] * x[2] + 0.02 * x[1] * x[3]
+ 0.34 * x[2]^2 + 0.07 * x[2] * x[3] + 0.48 * x[3]^2)) )
hin <- function(x) c( x[1] + 1, x[2] + 1, x[3] + 1, -x[1] + 1, -x[2] + 1,
-x[3] + 1)
gr <- function(x) nl.grad(x, fn)
hinjac <- function(x) nl.jacobian(x, hin)
auglag(x0, fn, gr = NULL, hin = hin, localsolver = "SLSQP")
#####
##### END PART 1 #####

##### PART 2: DESIRABILITY ANALYSIS #####
#####
# This R Code determines x_opt and the corresponding Overall DF
# The proportion of impurities, its variance and the estimate of MSE are
# obtained by introducing x_opt in the adjusted models for
# Y_Mean , Y_Var and Y_MSE
#####
# install.packages("desirability") # Install this package if not
# already installed

library(desirability)
### Fitted Quadratic Response Surface Models
MeanPred <- function(x) (14.80 - 8.17 * x[1] - 9.09 * x[2] - 0.14 * x[3]
+ 0.52 * x[1]^2 + 8.30 * x[1] * x[2] + 0.07 * x[1] * x[3]
+ 5.01 * x[2]^2 + 0.18 * x[2] * x[3] + 0.18 * x[3]^2)
VariancePred <- function(x) (exp(2.59 + 0.03 * x[1] - 1.66 * x[2] + 0.58 * x[3]
- 0.21 * x[1]^2 + 0.03 * x[1] * x[2] + 0.02 * x[1] * x[3]
+ 0.34 * x[2]^2 + 0.07 * x[2] * x[3] + 0.48 * x[3]^2))
#### Defining individual desirability functions
MeanD <- dMin(7.11, 46.5)
VarianceD <- dMin(2.21, 79.04)
### Maximizing Desirability
##### The penalty approach
rsmOpt <- function(x, dObject, space = "square")
{
  impurity <- MeanPred(x)
  variability <- VariancePred(x)
  out <- predict(dObject, data.frame(impurity = impurity, variability = variability))
  if(space == "circular")
  {
    if(sqrt(sum(x^2)) > 1) out <- 0
  } else if(space == "square") if(any(abs(x) > 1)) out <- 0
  out
}
##### The Nelder-Mead simplex method
searchGrid <- expand.grid(Reaction_Temperature = seq(-1, 1, length = 5),
```

D.2. R Code for Desirability Analysis (Chemical process problem)

```
Catalyst_Concentration = seq(-1, 1, length = 5) ,
Reagent_B = seq(-1, 1, length = 5))
for(i in 1:dim(searchGrid)[1])
{
tmp <- optim(as.vector(searchGrid[i,]), rsmOpt, dObject = overallD,
space = "square", control = list(fnscale = -1))
if(i == 1)
{
best <- tmp
} else {
if(tmp$value > best$value)
best <- tmp
}
}
print(best,digits = 5)
##### x_opt #####
x1 = 1.0000
x2 = 0.43189
x3 = -0.57353
##### Estimated Values #####
#### for MEAN, VARIANCE and MSE
MeanPred <- (14.80 - 8.17 * x1 - 9.09 * x2 - 0.14 * x3 + 0.52 * x1^2
+ 8.30 * x1 * x2 + 0.07 * x1 * x3 + 5.01 * x2^2 + 0.18 * x2 * x3
+ 0.18 * x3^2)
VariancePred <- (exp(2.59 + 0.03 * x1 - 1.66 * x2 + 0.58 * x3 - 0.21 * x1^2
+ 0.03 * x1 * x2 + 0.02 * x1 * x3 + 0.34 * x2^2 + 0.07 * x2 * x3
+ 0.48 * x3^2))
MSE_DOAD <- ((14.80 - 8.17 * x1 - 9.09 * x2 - 0.14 * x3 + 0.52 * x1^2
+ 8.30 * x1 * x2 + 0.07 * x1 * x3 + 5.01 * x2^2 + 0.18 * x2 * x3
+ 0.18 * x3^2)^2 +(exp(2.59 + 0.03 * x1 - 1.66 * x2 + 0.58 * x3
- 0.21 * x1^2 + 0.03 * x1 * x2 + 0.02 * x1 * x3 + 0.34 * x2^2
+ 0.07 * x2 * x3 + 0.48 * x3^2)))
print(MeanPred,digits = 3)
print(VariancePred,digits = 3)
print(MSE_DOAD,digits = 3)
##### END PART 2 #####

##### PART 3: GRAPHS #####
#####
# This R Code plots Individual Desirability Functions for the proportion
# of impurities (MEAN) and its variance
#####
par(oma = c(0, 0, 2, 0))
par(mfrow = c(1, 2))
##### MEAN #####
par(mar = c(4.3, 4.1, 2.1, 1.1))
MeanD1 <- dMin(7.11, 46.5)
plot(MeanD1, col = "blue", lwd = 2)
abline(v = 7.8, col = "magenta", lty = 11, lwd = 2)
abline(h = 0.97446, col = "magenta", lty = 11, lwd = 2)
text(40, .7, "MEAN", cex=1.2, col = "black")
arrows(20,0.85,7.8,0.97446, length=0.1, pch = 2, lwd=3)
text(30, 0.85, "(7.8,0.97446)", cex=1.2, col = "blue")
```

D.3. R Code for DF based on MSE criterion (Elastic element problem)

```
##### VARIANCE #####
par(mar = c(4.3, 2.1, 2.1, 1.1))
VarianceD1 <- dMin(2.21, 79.04)
plot(VarianceD1, col = "blue", lwd = 2)
abline(v = 4.79, col = "magenta", lty = 11, lwd = 2)
abline(h = 0.97446, col = "magenta", lty = 11, lwd = 2)
text(60, .7, "VARIANCE", cex = 1.2, col = "black")
arrows(20, 0.85, 4.79, 0.97446, length = 0.1, pch = 2, lwd = 3)
text(39, 0.85, "(4.79, 0.97446)", cex = 1.2, col = "blue")
title(main = "Desirability function plot", sub = "Nothing", col.main = "blue",
outer = T)
##### END PART 3 #####
```

D.3 R Code for DF based on MSE criterion (Elastic element problem)

```
##### PART 1 #####
#####
# This R Code determines x_opt and the corresponding MSE for Non-linearity
# and Hysteresis. Estimated values for Non-linearity and its variance,
# Hysteresis and its variance, are obtained by plugging x_opt into the
# corresponding adjusted models
#####
##### MINIMIZING MSE FOR y_1
rm(list=ls())
library("nloptr")
x0 <- c( 1, 1, 1 )
fn <- function(x) (1.377 - 0.361 * x[1] - 0.155 * x[2] + 0.077 * x[3]
- 0.148 * x[1] * x[2] + 0.022 * x[1] * x[3] + 0.013 * x[2] * x[3]
+ 0.042 * x[1]^2 + 0.007 * x[2]^2 + 0.002 * x[3]^2 - 1)^2
+ (3.925*10^(-3) - 9.88 * 10^(-4) * x[1] + 6.84 * 10^(-4) * x[2]
- 6.62 * 10^(-4) * x[3] + 1.64 * 10^(-4) * x[1]^2 + 3.7 * 10^(-5) * x[2]^2
+ 3.4 * 10^(-5) * x[3]^2 - 1.36 * 10^(-4) * x[1] * x[2]
+ 5.2 * 10^(-5) * x[1] * x[3] - 5.4 * 10^(-5) * x[2] * x[3])
hin <- function(x) c( x[1] + 1, x[2] + 1, x[3] + 1, -x[1] + 1, -x[2] + 1,
-x[3] + 1)
gr <- function(x) nl.grad(x, fn)
hinjac <- function(x) nl.jacobian(x, hin)
auglag(x0, fn, gr = NULL, hin = hin, localsolver = "SLSQP")
##### x_opt #####
x1 = 1.0000000
x2 = 0.5523737
x3 = 1.0000000
##### Estimated Values #####
##### MEAN, VARIANCE and MSE for y_1
NonlinearityPred <- (1.377 - 0.361 * x1 - 0.155 * x2 + 0.077 * x3
- 0.148 * x1 * x2 + 0.022 * x1 * x3 + 0.013 * x2 * x3
+ 0.042 * x1^2 + 0.007 * x2^2 + 0.002 * x3^2)
NonlinearityVariance <- (3.925 * 10^(-3) - 9.88 * 10^(-4) * x1
+ 6.84 * 10^(-4) * x2 - 6.62 * 10^(-4) * x3 + 1.64 * 10^(-4) * x1^2
```

D.3. R Code for DF based on MSE criterion (Elastic element problem)

```
+ 3.7 * 10^(-5) * x2^2 + 3.4 * 10^(-5) * x3^2 - 1.36 * 10^(-4) * x1 * x2
+ 5.2 * 10^(-5) * x1 * x3 - 5.4 * 10^(-5) * x2 * x3)
NonlinearityMSE <- ((1.377 - 0.361 * x1 - 0.155 * x2 + 0.077 * x3
- 0.148 * x1 * x2 + 0.022 * x1 * x3 + 0.013 * x2 * x3
+ 0.042 * x1^2 + 0.007 * x2^2 + 0.002 * x3^2 - 1)^2
+ ( 3.925*10^(-3)-9.88*10^(-4)*x1+6.84*10^(-4)*x2
-6.62*10^(-4)*x3+1.64*10^(-4)*x1^2+3.7*10^(-5)*x2^2
+3.4*10^(-5)*x3^2-1.36*10^(-4)*x1*x2
+5.2*10^(-5)*x1*x3-5.4*10^(-5)*x2*x3))
print(NonlinearityPred, digits = 4)
print(NonlinearityVariance, digits = 4)
print(NonlinearityMSE, digits = 4)

##### MAXIMIZING MSE FOR y_1
rm(list=ls())
library("nloptr")
x0 <- c( 0, 0, 0 )
fn <- function(x) -((1.377 - 0.361 * x[1] - 0.155 * x[2] + 0.077 * x[3]
- 0.148 * x[1] * x[2] + 0.022 * x[1] * x[3] + 0.013 * x[2] * x[3]
+ 0.042 * x[1]^2 + 0.007 * x[2]^2 + 0.002 * x[3]^2 - 1)^2
+ (3.925*10^(-3) - 9.88 * 10^(-4) * x[1] + 6.84 * 10^(-4) * x[2]
- 6.62 * 10^(-4) * x[3] + 1.64 * 10^(-4) * x[1]^2
+ 3.7 * 10^(-5) * x[2]^2 + 3.4 * 10^(-5) * x[3]^2
- 1.36 * 10^(-4) * x[1] * x[2] + 5.2 * 10^(-5) * x[1] * x[3]
- 5.4 * 10^(-5) * x[2] * x[3]))
hin <- function(x) c( x[1] + 1, x[2] + 1, x[3] + 1, -x[1] + 1, -x[2] + 1,
-x[3] + 1)
gr <- function(x) nl.grad(x, fn)
hinjac <- function(x) nl.jacobian(x, hin)
auglag(x0, fn, gr = NULL, hin = hin, localsolver = "lbfgs")
##### x_opt #####
x1 = -1
x2 = 1
x3 = 1
##### Estimated Values #####
##### MEAN, VARIANCE and MSE for y_1
NonlinearityPred <- (1.377 - 0.361 * x1 - 0.155 * x2 + 0.077 * x3
- 0.148 * x1 * x2 + 0.022 * x1 * x3 + 0.013 * x2 * x3 + 0.042 * x1^2
+ 0.007 * x2^2 + 0.002 * x3^2)
NonlinearityVariance <- (3.925 * 10^(-3)-9.88 * 10^(-4) * x1
+ 6.84 * 10^(-4) * x2 - 6.62 * 10^(-4) * x3 + 1.64 * 10^(-4) * x1^2
+ 3.7 * 10^(-5) * x2^2 + 3.4 * 10^(-5) * x3^2 - 1.36 * 10^(-4) * x1 * x2
+ 5.2 * 10^(-5) * x1 * x3 - 5.4 * 10^(-5) * x2 * x3)
NonlinearityMSE <- ((1.377 - 0.361 * x1 - 0.155 * x2 + 0.077 * x3
- 0.148 * x1 * x2 + 0.022 * x1 * x3 + 0.013 * x2 * x3 + 0.042 * x1^2
+ 0.007 * x2^2 + 0.002 * x3^2 - 1)^2 + ( 3.925 * 10^(-3) - 9.88 * 10^(-4) * x1
+ 6.84 * 10^(-4) * x2 - 6.62 * 10^(-4) * x3 + 1.64 * 10^(-4) * x1^2
+ 3.7 * 10^(-5) * x2^2 + 3.4 * 10^(-5) * x3^2 - 1.36 * 10^(-4) * x1 * x2
+ 5.2 * 10^(-5) * x1 * x3 - 5.4 * 10^(-5) * x2 * x3))
print(NonlinearityPred, digits = 4)
print(NonlinearityVariance, digits = 4)
print(NonlinearityMSE, digits = 4)
##### MINIMIZING MSE FOR y_2
```


D.3. R Code for DF based on MSE criterion (Elastic element problem)

```
rm(list=ls())
library("nloptr")
x0 <- c( 0, 0, 0 )
fn <- function(x) (1.660 + 0.592 * x[1] + 0.438 * x[2] - 0.095 * x[3]
+ 0.301 * x[1] * x[2] - 0.143 * x[1] * x[3]
- 0.033 * x[2] * x[3] + 0.247 * x[1]^2 - 0.123 * x[2]^2
+ 0.047 * x[3]^2)^2 + (0.04312 + 0.009 * x[1]
+ 1.032 * 10^(-3) * x[2] + 6.48 * 10^(-4) * x[3]
+ 0.00653 * x[1]^2 + 4.682 * 10^(-3) * x[2]^2
+ 2.12 * 10^(-4) * x[3]^2 - 6.972 * 10^(-3) * x[1] * x[2]
- 1.108 * 10^(-4) * x[1] * x[3] + 1.956 * 10^(-3) * x[2] * x[3])
hin <- function(x) c( x[1] + 1, x[2] + 1, x[3] + 1, -x[1] + 1, -x[2] + 1,
-x[3] + 1)
gr <- function(x) nl.grad(x, fn)
hinjac <- function(x) nl.jacobian(x, hin)
auglag(x0, fn, gr = NULL, hin = hin, localsolver = "lbfgs")
##### x_opt #####
x1 = -0.7194638
x2 = -1.0000000
x3 = -0.4274953
##### Estimated Values #####
##### MEAN, VARIANCE and MSE for y_2
HysteresisPred <- (1.660 + 0.592 * x1 + 0.438 * x2 - 0.095 * x3 + 0.301 * x1 * x2
- 0.143 * x1 * x3 - 0.033 * x2 * x3 + 0.247 * x1^2 - 0.123 * x2^2 + 0.047 * x3^2)
HysteresisVariance <- (0.04312 + 0.009 * x1 + 1.032 * 10^(-3) * x2
+ 6.48 * 10^(-4) * x3 + 0.00653 * x1^2 + 4.682 * 10^(-3) * x2^2
+ 2.12 * 10^(-4) * x3^2 - 6.972 * 10^(-3) * x1 * x2 - 1.108 * 10^(-4) * x1 * x3
+ 1.956 * 10^(-3) * x2 * x3)
HysteresisMSE <- ((1.660 + 0.592 * x1 + 0.438 * x2 - 0.095 * x3 + 0.301 * x1 * x2
- 0.143 * x1 * x3 - 0.033 * x2 * x3 + 0.247 * x1^2 - 0.123 * x2^2 + 0.047 * x3^2)^2
+ (0.04312 + 0.009 * x1 + 1.032 * 10^(-3) * x2 + 6.48 * 10^(-4) * x3
+ 0.00653 * x1^2 + 4.682 * 10^(-3) * x2^2 + 2.12 * 10^(-4) * x3^2 - 6.972 * 10^(-3)
* x1 * x2 - 1.108 * 10^(-4) * x1 * x3 + 1.956 * 10^(-3) * x2 * x3))
print(HysteresisPred, digits = 3)
print(HysteresisVariance, digits = 3)
print(HysteresisMSE, digits = 3)
##### MAXIMIZING MSE FOR y_2
rm(list=ls())
library("nloptr")
x0 <- c( 1, 1, 1 )
fn <- function(x) -((1.660 + 0.592 * x[1] + 0.438 * x[2] - 0.095 * x[3]
+ 0.301 * x[1] * x[2] - 0.143 * x[1] * x[3]
- 0.033 * x[2] * x[3] + 0.247 * x[1]^2 - 0.123 * x[2]^2
+ 0.047 * x[3]^2)^2 + (0.04312 + 0.009 * x[1]
+ 1.032 * 10^(-3) * x[2] + 6.48 * 10^(-4) * x[3]
+ 0.00653 * x[1]^2 + 4.682 * 10^(-3) * x[2]^2
+ 2.12 * 10^(-4) * x[3]^2 - 6.972 * 10^(-3) * x[1] * x[2]
- 1.108 * 10^(-4) * x[1] * x[3] + 1.956 * 10^(-3) * x[2] * x[3]))
hin <- function(x) c( x[1] + 1, x[2] + 1, x[3] + 1, -x[1] + 1, -x[2] + 1,
-x[3] + 1)
gr <- function(x) nl.grad(x, fn)
hinjac <- function(x) nl.jacobian(x, hin)
auglag(x0, fn, gr = NULL, hin = hin, localsolver = "lbfgs")
```

D.3. R Code for DF based on MSE criterion (Elastic element problem)

```
##### x_opt #####
x1 = 1
x2 = 1
x3 = 1
##### Estimated Values #####
##### MEAN, VARIANCE and MSE for y_2
HysteresisPred <- (1.660 + 0.592 * x1 + 0.438 * x2 - 0.095 * x3 + 0.301 * x1 * x2
- 0.143 * x1 * x3 - 0.033 * x2 * x3 + 0.247 * x1^2 - 0.123 * x2^2 + 0.047 * x3^2)
HysteresisVariance <- (0.04312 + 0.009 * x1 + 1.032 * 10^(-3) * x2
+ 6.48 * 10^(-4) * x3 + 0.00653 * x1^2 + 4.682 * 10^(-3) * x2^2
+ 2.12 * 10^(-4) * x3^2 - 6.972 * 10^(-3) * x1 * x2 - 1.108 * 10^(-4) * x1 * x3
+ 1.956 * 10^(-3) * x2 * x3)
HysteresisMSE <- ((1.660 + 0.592 * x1 + 0.438 * x2 - 0.095 * x3 + 0.301 * x1 * x2
- 0.143 * x1 * x3 - 0.033 * x2 * x3 + 0.247 * x1^2 - 0.123 * x2^2 + 0.047 * x3^2)^2
+ (0.04312 + 0.009 * x1 + 1.032 * 10^(-3) * x2 + 6.48 * 10^(-4) * x3
+ 0.00653 * x1^2 + 4.682 * 10^(-3) * x2^2 + 2.12 * 10^(-4) * x3^2 - 6.972 * 10^(-3)
* x1 * x2 - 1.108 * 10^(-4) * x1 * x3 + 1.956 * 10^(-3) * x2 * x3))
print(HysteresisPred, digits = 3)
print(HysteresisVariance, digits = 3)
print(HysteresisMSE, digits = 3)
##### END PART 1 #####

#####
# This R Code graphs the Contour plot and Perspective plot
# for the Mean Square Error for Non-linearity and Hysteresis
#####

##### PART 2: MSE FOR NON-LINEARITY #####
##### x_2 fixed
par(oma = c(0, 0, 2, 0))
par(mfrow = c(1, 2))
par(mar = c(4.1, 4.1, 2.1, 1.1))
x_1 <- seq(-1, 1, len = 100)
x_2 <- -1
x_3 <- seq(-1, 1, len = 100)
f <- outer(x_1, x_3, function(x_1, x_3) (1.377 - 0.361 * x_1 - 0.155 * x_2
+ 0.077 * x_3 - 0.148 * x_1 * x_2 + 0.022 * x_1 * x_3 + 0.013 * x_2 * x_3
+ 0.042 * x_1^2 + 0.007 * x_2^2 + 0.002 * x_3^2 - 1)^2 + (3.925 * 10^(-3)
- 9.88 * 10^(-4) * x_1
+ 6.84 * 10^(-4) * x_2 - 6.62 * 10^(-4) * x_3 + 1.64 * 10^(-4) * x_1^2
+ 3.7 * 10^(-5) * x_2^2 + 3.4 * 10^(-5) * x_3^2 - 1.36 * 10^(-4) * x_1 * x_2
+ 5.2 * 10^(-5) * x_1 * x_3 - 5.4 * 10^(-5) * x_2 * x_3))
contour(x_1, x_3, f, levels = c(0.005, 0.025, 0.05, 0.1, 0.158, 0.2, 0.3
, 0.4, 0.5, 0.6, 0.7),
xlab=expression(x[1]), ylab=expression(x[3]), main= "x2 fixed", col="blue")
arrows(0, -0.5, 0.38916, -0.98629, length = 0.1, pch = 2, lwd = 3)
text(0.1, -0.45, "0.158", cex = 1.2, col = "red")

par(mar=c(3.1,0.1,2.1,0.1))
y<-function(x_1,x_3){ (1.377 - 0.361 * x_1 - 0.155 * x_2 + 0.077 * x_3
- 0.148 * x_1 * x_2 + 0.022 * x_1 * x_3 + 0.013 * x_2 * x_3 + 0.042 * x_1^2
+ 0.007 * x_2^2 + 0.002 * x_3^2 - 1)^2 + (3.925 * 10^(-3) - 9.88 * 10^(-4) * x_1
```

D.3. R Code for DF based on MSE criterion (Elastic element problem)

```
+ 6.84 * 10(-4) * x_2 - 6.62 * 10(-4) * x_3 + 1.64 * 10(-4) * x_12
+ 3.7 * 10(-5) * x_22 + 3.4 * 10(-5) * x_32 - 1.36 * 10(-4) * x_1 * x_2
+ 5.2 * 10(-5) * x_1 * x_3 - 5.4 * 10(-5) * x_2 * x_3
}
f<-outer(x_1,x_3,y)
persp(x_1,x_3,f,theta =-40, phi = 20,col = rainbow(50), xlab = expression(x_1),
ylab = expression(x_3), zlab =expression(MSE) ,main= "x2 fixed")
title(main = "Contour and Perspective plots (MSE for Non-linearity)",
sub = "Nothing", col.main="blue", outer = T)
##### END PART 2 #####

##### PART 3: MSE FOR HYSTERESIS #####
rm(list=ls())
#install.packages("rsm") # Install this package if not already installed
library("rsm")
#####
##### x_2 fixed
par(oma = c(0, 0, 2, 0))
par(mfrow = c(1, 2))
par(mar = c(4.1, 4.1, 2.1, 1.1))
x_1 <- seq(-1,1,len=100)
x_2 <- -1
x_3 <- seq(-1,1,len=100)
f <- outer(x_1, x_3, function(x_1, x_3) (1.660 + 0.592 * x_1 + 0.438 * x_2
- 0.095 * x_3 + 0.301 * x_1 * x_2 - 0.143 * x_1 * x_3 - 0.033 * x_2 * x_3
+ 0.247 * x_12 - 0.123 * x_22 + 0.047 * x_32)2 + (0.04312 + 0.009 * x_1
+ 1.032 * 10(-3) * x_2 + 6.48 * 10(-4) * x_3 + 0.00653 * x_12
+ 4.682 * 10(-3) * x_22 + 2.12 * 10(-4) * x_32 - 6.972 * 10(-3) * x_1 * x_2
- 1.108 * 10(-4) * x_1 * x_3 + 1.956 * 10(-3) * x_2 * x_3))
contour(x_1, x_3, f,levels = c(1.06, 1.1, 1.15, 1.2, 1.3, 1.5, 1.75, 2.05, 3),
xlab = expression(x[1]), ylab = expression(x[3]),main = "x2 fixed",
col = "blue")
arrows(0, -0.5, 0.38916,-0.98629, length=0.1, pch = 2, lwd=3)
text(0.1, -0.45, "2.05", cex=1.2, col = "red")

par(mar=c(3.1,0.1,2.1,0.1))
y<-function(x_1,x_3){ (1.660 + 0.592 * x_1 + 0.438 * x_2 - 0.095 * x_3
+ 0.301 * x_1 * x_2 - 0.143 * x_1 * x_3 - 0.033 * x_2 * x_3 + 0.247 * x_12
- 0.123 * x_22 + 0.047 * x_32)2 + (0.04312 + 0.009 * x_1 + 1.032 * 10(-3) * x_2
+ 6.48 * 10(-4) * x_3 + 0.00653 * x_12 + 4.682 * 10(-3) * x_22 + 2.12 * 10(-4)
* x_32 - 6.972 * 10(-3) * x_1 * x_2 - 1.108 * 10(-4) * x_1 * x_3
+ 1.956 * 10(-3) * x_2 * x_3)
}
f<-outer(x_1,x_3,y)
persp(x_1,x_3,f,theta =-40, phi = 20,col = rainbow(50), xlab = expression(x_1),
ylab = expression(x_3), zlab =expression(MSE) ,main= "x2 fixed")
title(main = "Contour and Perspective plots (MSE for Hysteresis)", sub = "Nothing",
col.main="blue", outer = T)
##### END PART 3 #####

##### PART 4: DESIRABILITY ANALYSIS #####
#####
# This R Code determines x_opt and the corresponding Overall DF
```

D.3. R Code for DF based on MSE criterion (Elastic element problem)

```
# Optimal values for Non-linearity and Hysteresis are obtained
# by introducing x_opt in their adjusted models
#####
# install.packages("desirability") # Install this package if not
# already installed

library(desirability)
### Fitted Quadratic Response Surface Models
NonlinearityPred <- function(x) (1.377 - 0.361 * x[1] - 0.155 * x[2]
+ 0.077 * x[3] - 0.148 * x[1] * x[2] + 0.022 * x[1] * x[3] + 0.013 * x[2]
* x[3] + 0.042 * x[1]^2 + 0.007 * x[2]^2 + 0.002 * x[3]^2 - 1)^2 +
(3.925 * 10^(-3) - 9.88 * 10^(-4) * x[1] + 6.84 * 10^(-4) * x[2] - 6.62 * 10^(-4)
* x[3] + 1.64 * 10^(-4) * x[1]^2 + 3.7 * 10^(-5) * x[2]^2 + 3.4 * 10^(-5) * x[3]^2
- 1.36 * 10^(-4) * x[1] * x[2] + 5.2 * 10^(-5) * x[1] * x[3] - 5.4 * 10^(-5) * x[2]
* x[3])
HysteresisPred <- function(x) (1.660 + 0.592 * x[1] + 0.438 * x[2]
- 0.095 * x[3] + 0.301 * x[1] * x[2] - 0.143 * x[1] * x[3] - 0.033 * x[2] * x[3]
+ 0.247 * x[1]^2 - 0.123 * x[2]^2 + 0.047 * x[3]^2)^2 + (0.04312
+ 0.009 * x[1] + 1.032 * 10^(-3) * x[2] + 6.48 * 10^(-4) * x[3]
+ 0.00653 * x[1]^2 + 4.682 * 10^(-3) * x[2]^2 + 2.12 * 10^(-4) * x[3]^2
- 6.972 * 10^(-3) * x[1] * x[2] - 1.108 * 10^(-4) * x[1] * x[3]
+ 1.956 * 10^(-3) * x[2] * x[3]) )
#### Defining individual desirability functions
NonlinearityD <- dMin(0.00281, 0.7277)
HysteresisD <- dMin(1.06, 8.42)
### Maximizing Desirability
##### The penalty approach
rsmOpt <- function(x, dObject, space = "square")
{
  Nonlinearity <- NonlinearityPred(x)
  Hysteresis <- HysteresisPred(x)
  out <- predict(dObject, data.frame(Nonlinearity = Nonlinearity,
                                     Hysteresis = Hysteresis))

  if(space == "circular")
  {
    if(sqrt(sum(x^2)) > 1) out <- 0
  } else if(space == "square") if(any(abs(x) > 1)) out <- 0
  out
}
##### The Nelder-Mead simplex method
searchGrid <- expand.grid(LozengeAngle = seq(-1, 1, length = 5),
  BoreDiameter = seq(-1, 1, length = 5),
  HalfLength = seq(-1, 1, length = 5))
for(i in 1:dim(searchGrid)[1])
{
  tmp <- optim(as.vector(searchGrid[i,]), rsmOpt, dObject = overallD,
space = "square", control = list(fnscale = -1))
  if(i == 1)
  {
    best <- tmp
  } else {
    if(tmp$value > best$value)
    best <- tmp
  }
}
```

D.3. R Code for DF based on MSE criterion (Elastic element problem)

```
}
print(best, digits = 7)
##### x_opt #####
x1 = 0.3891627
x2 = -1.0000000
x3 = -0.9862911
##### Estimated Values #####
##### MEAN, VARIANCE and MSE for y_1
#####
NonlinearityPred <- (1.377 - 0.361 * x1 - 0.155 * x2 + 0.077 * x3
- 0.148 * x1 * x2 + 0.022 * x1 * x3 + 0.013 * x2 * x3 + 0.042 * x1^2
+ 0.007 * x2^2 + 0.002 * x3^2)
NonlinearityVariance <- (3.925 * 10^(-3) - 9.88 * 10^(-4) * x1 + 6.84 * 10^(-4)
* x2 - 6.62 * 10^(-4) * x3 + 1.64 * 10^(-4) * x1^2 + 3.7 * 10^(-5) * x2^2
+ 3.4 * 10^(-5) * x3^2 - 1.36 * 10^(-4) * x1 * x2 + 5.2 * 10^(-5) * x1 * x3
- 5.4 * 10^(-5) * x2 * x3)
NonlinearityMSE <- ((1.377 - 0.361 * x1 - 0.155 * x2 + 0.077 * x3
- 0.148 * x1 * x2 + 0.022 * x1 * x3 + 0.013 * x2 * x3
+ 0.042 * x1^2 + 0.007 * x2^2 + 0.002 * x3^2 - 1)^2 +
( 3.925 * 10^(-3) - 9.88 * 10^(-4) * x1 + 6.84 * 10^(-4) * x2
- 6.62 * 10^(-4) * x3 + 1.64 * 10^(-4) * x1^2 + 3.7 * 10^(-5) * x2^2
+ 3.4 * 10^(-5) * x3^2 - 1.36 * 10^(-4) * x1 * x2 + 5.2 * 10^(-5) * x1 * x3
- 5.4 * 10^(-5) * x2 * x3))
print(NonlinearityPred, digits = 5)
print(NonlinearityVariance, digits = 5)
print(NonlinearityMSE, digits = 5)

##### MEAN, VARIANCE and MSE for y_2
#####
HysteresisPred <- (1.660 + 0.592 * x1 + 0.438 * x2 - 0.095 * x3
+ 0.301 * x1 * x2 - 0.143 * x1 * x3 - 0.033 * x2 * x3
+ 0.247 * x1^2 - 0.123 * x2^2 + 0.047 * x3^2)
HysteresisVariance <- (0.04312 + 0.009 * x1 + 1.032 * 10^(-3) * x2
+ 6.48 * 10^(-4) * x3 + 0.00653 * x1^2 + 4.682 * 10^(-3) * x2^2
+ 2.12 * 10^(-4) * x3^2 - 6.972 * 10^(-3) * x1 * x2
- 1.108 * 10^(-4) * x1 * x3 + 1.956 * 10^(-3) * x2 * x3)
HysteresisMSE <- ((1.660 + 0.592 * x1 + 0.438 * x2 - 0.095 * x3
+ 0.301 * x1 * x2 - 0.143 * x1 * x3 - 0.033 * x2 * x3
+ 0.247 * x1^2 - 0.123 * x2^2 + 0.047 * x3^2)^2
+ (0.04312 + 0.009 * x1 + 1.032 * 10^(-3) * x2
+ 6.48 * 10^(-4) * x3 + 0.00653 * x1^2
+ 4.682 * 10^(-3) * x2^2 + 2.12 * 10^(-4) * x3^2
- 6.972 * 10^(-3) * x1 * x2 - 1.108 * 10^(-4) * x1 * x3
+ 1.956 * 10^(-3) * x2 * x3))
print(HysteresisPred, digits = 5)
print(HysteresisVariance, digits = 5)
print(HysteresisMSE, digits = 5)
##### END PART 4 #####

##### PART 5: DESIRABILITY GRAPHS #####
#####
# This R Code plots Individual Desirability Functions
# of MSE functions for Non-linearity and Hysteresis
```

D.4. R Code for simulating SNR and determining powers and test sizes

```
#####
par(mfrow=c(1,2))
##### MSE_Non_linearity
par(mar = c(4.3, 4.1, 2.1, 1.1))
Non_linearity_D <- dMin(0.04, 0.7277, scale = c(1))
plot(Non_linearity_D, col = "red", lwd = 3)
abline(v = 0.158, col = "magenta", lty = 11, lwd = 2)
abline(h = 0.82496, col = "magenta", lty = 11, lwd = 2)
text(.5, .97, "MSE for Non-linearity", cex = 1.2, col = "black")
arrows(0.3,0.75,0.158,0.82496, length = 0.1, pch = 2, lwd = 3)
text(0.5, 0.72, "(0.158,0.82496)", cex=1.2, col = "blue")
##### MSE_Hysteresis
par(mar = c(4.3, 2.1, 2.1, 1.1))
Hysteresis_D <- dMin(0.7, 8.42, scale = c(1))
plot(Hysteresis_D, col = "red",lwd=3)
abline(v = 2.05, col = "magenta", lty = 11, lwd = 2)
abline(h = 0.82496, col = "magenta", lty = 11, lwd = 2)
text(6, .97, "MSE for Hysteresis", cex =1.2, col = "black")
arrows(3.8,0.75,2.05,0.82496, length = 0.2, pch = 2, lwd = 3)
text(5.2, 0.72, "(2.05,0.82496)", cex = 1.2, col = "blue")
title(main = "Plot for the Desirability Function", sub = "Nothing",
col.main = "blue",
outer = T)
##### END PART 5 #####
```

D.4 R Code for simulating SNR and determining powers and test sizes

```
#####
## This R Code determines test sizes when simulations are conducted
## H_0, this is, with equal parameters, otherwise determines powers
#####
##### PARAMETERS #####
rm(list = ls())
set.seed(10)
nx # Sample size for simulating samples from Normal population x
ny # Sample size for simulating samples from Normal population y
mu.x # Population mean of the Normal population x
mu.y # Population mean of the Normal population y
sig.x # Population standard deviation of the Normal population x
sig.y # Population standard deviation of the Normal population y
alpha # Test size
n.sim # Number of simulations
m # Rate of increment in determining various sample sizes
u # Starting value in determining various sample sizes for SNR
#####
##### PART 1: SNR_T #####
##### SIGNAL-TO-NOISE RATIO FOR THE NOMINAL-THE-BEST CASE #####
calc.pvalue.t <- function(nx, ny, sig.SNR.x, sig.SNR.y, mu.SNR.x.hat,
mu.SNR.y.hat, alpha){
  x <- rnorm(nx, mu.x, sig.x)
```

D.4. R Code for simulating SNR and determining powers and test sizes

```
y <- rnorm(ny, mu.y, sig.y)

x.bar = mean(x)
x.sig.square = var(x)

y.bar = mean(y)
y.sig.square = var(y)

SNR.x.hat = 10 * log(x.bar^2/x.sig.square)
mu.SNR.x.hat = 10 * log(mu.x^2/sig.x^2)
sig.SNR.x.hat = (10/log(10))^2*((4 * sig.x^2)/(nx^2 * mu.x^2) + (2/nx^2))

SNR.y.hat = 10 * log(y.bar^2/y.sig.square)
mu.SNR.y.hat = 10 * log(mu.y^2/sig.y^2)
sig.SNR.y.hat = (10/log(10))^2*((4 * sig.y^2)/(ny^2 * mu.y^2) + (2/ny^2))

# n.SNR.x = nx
# n.SNR.y = ny
SNR.x.hat = rnorm(nx, mu.SNR.x.hat, sig.SNR.x.hat)
SNR.y.hat = rnorm(ny, mu.SNR.y.hat, sig.SNR.y.hat)

t_test = t.test(SNR.x.hat, SNR.y.hat, alternative = "two.sided",
                paired = FALSE, conf.level = alpha)$p.value
if(t_test < alpha)(reject = 1) else(reject = 0)
return(reject)
}
calc.t<-function(nx, ny, mu.SNR.x.hat, mu.SNR.y.hat, sig.SNR.x, sig.SNR.y,
n.sim, alpha){
t_sum = 0
for (j in 1:n.sim){
t_temp = calc.pvalue.t(nx, ny, sig.SNR.x, sig.SNR.y, mu.SNR.x.hat,
mu.SNR.y.hat, alpha)
t_sum = t_temp + t_sum
}
p.value = t_sum/n.sim
return(p.value)
}
powers = matrix(NA,nrow = m, ncol = 2)
for (i in 1:m){
t.test.m = calc.t(nx = round(u + exp(i), -1), ny = round(u + exp(i), -1),
mu.SNR.x.hat, mu.SNR.y.hat,
sig.SNR.x, sig.SNR.y,n.sim, alpha)
powers[i,1] = round(u + exp(i), -1)
powers[i,2] = t.test.m
}
}
print(powers)
##### END PART 1 #####
##### PART 2: SNR_S #####
##### SIGNAL-TO-NOISE RATIO FOR THE SMALLER-THE-BETTER CASE #####
#####
calc.pvalue.t <- function(nx, ny, sig.SNR.x, sig.SNR.y, mu.SNR.x.hat,
mu.SNR.y.hat, alpha){
x <- rnorm(nx, mu.x, sig.x)
```

D.4. R Code for simulating SNR and determining powers and test sizes

```
y <- rnorm(ny, mu.y, sig.y)

x.bar = mean(x)
x.sig.square = var(x)

y.bar = mean(y)
y.sig.square = var(y)

SNR.x.hat = (-10/log(10)) * log(x.sig.square + x.bar^2)
mu.SNR.x.hat = (-10/log(10)) * log(sig.x^2 + mu.x^2)
sig.SNR.x.hat = (-10/log(10))^2*((sig.x^2/(nx^2 * (sig.x^2 + mu.x^2)^2))
+((8 * mu.x^2 * sig.x^4)/(nx^2 * (sig.x^2 + mu.x^2)^2)))

SNR.y.hat = (-10/log(10)) * log(y.sig.square + y.bar^2)
mu.SNR.y.hat = (-10/log(10)) * log(sig.y^2 + mu.y^2)
sig.SNR.y.hat = (-10/log(10))^2*((sig.y^2/(ny^2 * (sig.y^2 + mu.y^2)^2))
+((8 * mu.y^2 * sig.y^4)/(ny^2 * (sig.y^2 + mu.y^2)^2)))
# n.SNR.x = nx
# n.SNR.y = ny
SNR.x.hat = rnorm(nx, mu.SNR.x.hat, sig.SNR.x.hat)
SNR.y.hat = rnorm(ny, mu.SNR.y.hat, sig.SNR.y.hat)

t_test = t.test(SNR.x.hat, SNR.y.hat, alternative = "two.sided",
                paired = FALSE, conf.level = 1 - alpha)$p.value
if(t_test < alpha)(reject = 1) else(reject = 0)
return(reject)
}
calc.t<-function(nx, ny, mu.SNR.x.hat, mu.SNR.y.hat, sig.SNR.x, sig.SNR.y,
n.sim, alpha){
t_sum = 0
for (j in 1:n.sim){
t_temp = calc.pvalue.t(nx, ny, sig.SNR.x, sig.SNR.y, mu.SNR.x.hat,
mu.SNR.y.hat, alpha)
t_sum = t_temp + t_sum
}
p.value = t_sum/n.sim
return(p.value)
}
powers = matrix(NA,nrow = m, ncol = 2)
for (i in 1:m){
t.test.m = calc.t(nx = round(u + exp(i), -1), ny = round(u + exp(i), -1),
mu.SNR.x.hat, mu.SNR.y.hat, sig.SNR.x,
sig.SNR.y,n.sim, alpha)
powers[i,1] = round(u + exp(i), -1)
powers[i,2] = t.test.m
}
}
print(powers)
##### END PART 2 #####

##### PART 3: SNR_L #####
##### SIGNAL-TO-NOISE RATIO FOR THE LARGER-THE-BETTER CASE #####
#####
calc.pvalue.t <- function(nx, ny, sig.SNR.x, sig.SNR.y, mu.SNR.x.hat,
```


D.4. R Code for simulating SNR and determining powers and test sizes

```
mu.SNR.y.hat, alpha){
  x1 <- rnorm(nx, mu.x, sig.x)
  y1 <- rnorm(ny, mu.y, sig.y)

  x = 1/x1
  y = 1/y1

  x.bar = mean(x)
  x.sig.square = var(x)

  y.bar = mean(y)
  y.sig.square = var(y)

  SNR.x.hat = (-10/log(10)) * log(x.sig.square + x.bar^2)
  mu.SNR.x.hat = (-10/log(10)) * log(sig.x^2 + mu.x^2)
  sig.SNR.x.hat = (-10/log(10))^2*((sig.x^2/(nx^2 * (sig.x^2 + mu.x^2)^2))
    +((8 * mu.x^2 * sig.x^4)/(nx^2 * (sig.x^2 + mu.x^2)^2)))

  SNR.y.hat = (-10/log(10)) * log(y.sig.square + y.bar^2)
  mu.SNR.y.hat = (-10/log(10)) * log(sig.y^2 + mu.y^2)
  sig.SNR.y.hat = (-10/log(10))^2*((sig.y^2/(ny^2 * (sig.y^2 + mu.y^2)^2))
    +((8 * mu.y^2 * sig.y^4)/(ny^2 * (sig.y^2 + mu.y^2)^2)))
  # n.SNR.x = nx
  # n.SNR.y = ny
  SNR.x.hat = rnorm(nx, mu.SNR.x.hat, sig.SNR.x.hat)
  SNR.y.hat = rnorm(ny, mu.SNR.y.hat, sig.SNR.y.hat)

  t_test = t.test(SNR.x.hat, SNR.y.hat, alternative = "two.sided",
    paired = FALSE, conf.level = 1- alpha)$p.value
  if(t_test < alpha)(reject = 1) else(reject = 0)
  return(reject)
}
calc.t<-function(nx, ny, mu.SNR.x.hat, mu.SNR.y.hat, sig.SNR.x, sig.SNR.y,
n.sim, alpha){
  t_sum = 0
  for (j in 1:n.sim){
    t_temp = calc.pvalue.t(nx, ny, sig.SNR.x, sig.SNR.y, mu.SNR.x.hat, mu.SNR.
y.hat, alpha)
    t_sum = t_temp + t_sum
  }
  p.value = t_sum/n.sim
  return(p.value)
}
powers = matrix(NA,nrow = m, ncol = 2)
for (i in 1:m){
  t.test.m = calc.t(nx = round(u + exp(i), -1), ny = round(u + exp(i), -1),
mu.SNR.x.hat, mu.SNR.y.hat, sig.SNR.x,
sig.SNR.y, n.sim, alpha)
  powers[i,1] = round(u + exp(i), -1)
  powers[i,2] = t.test.m
}
print(powers)
##### END PART 3 #####
```